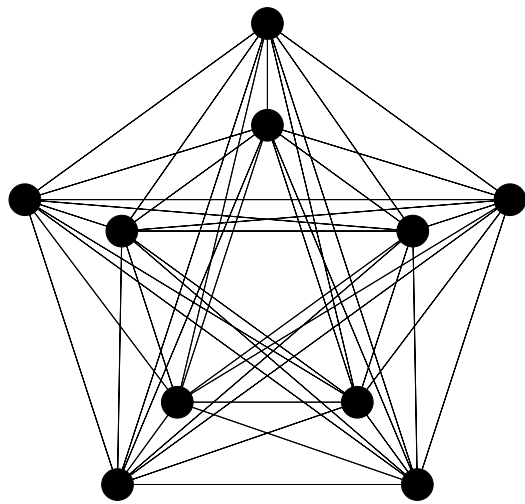
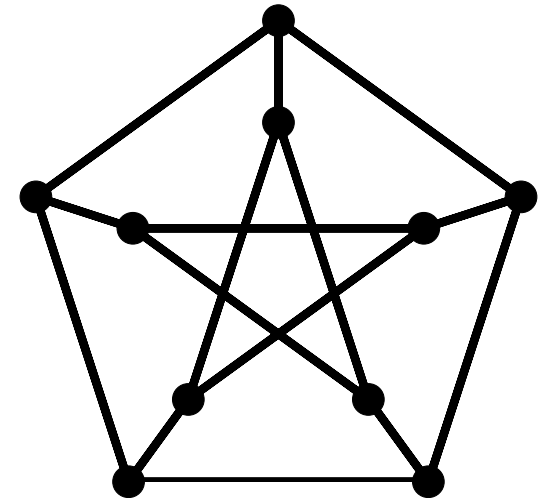
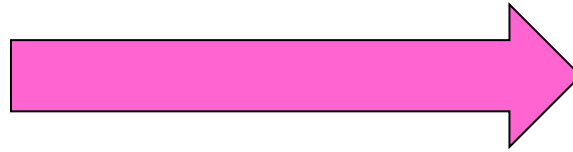


Spectral Sparsification of Graphs and Approximations of Matrices



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joint work with
Joshua Batson (MIT)
Nikhil Srivastava (IAS/Princeton)
Shang-Hua Teng (USC)

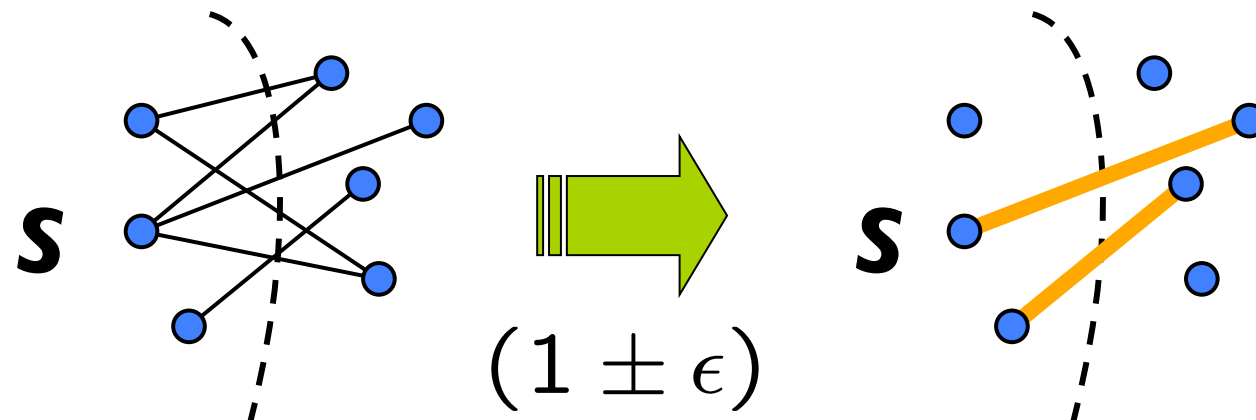
Weizmann, May 15, 2011

Objective of Sparsification:

Approximate any (weighted) graph by a sparse weighted graph.

Spanners - Preserve Distances [Chew '89]

Cut-Sparsifiers – preserve wt of edges leaving every set $S \subseteq V$ [Benczur-Karger '96]



Spectral Sparsification [S-Teng]

Approximate any (weighted) graph by a sparse weighted graph.

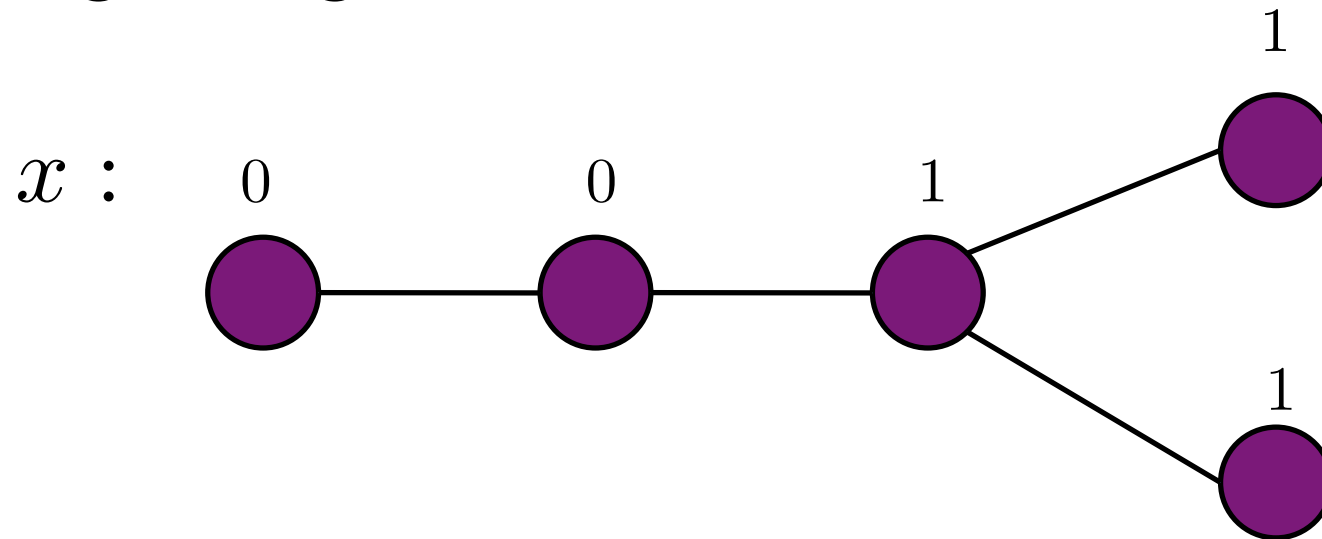
Graph  Quadratic Form

$$G = (V, E, w) \quad \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2$$
$$= x^T L_G x$$

L_G is Laplacian of G

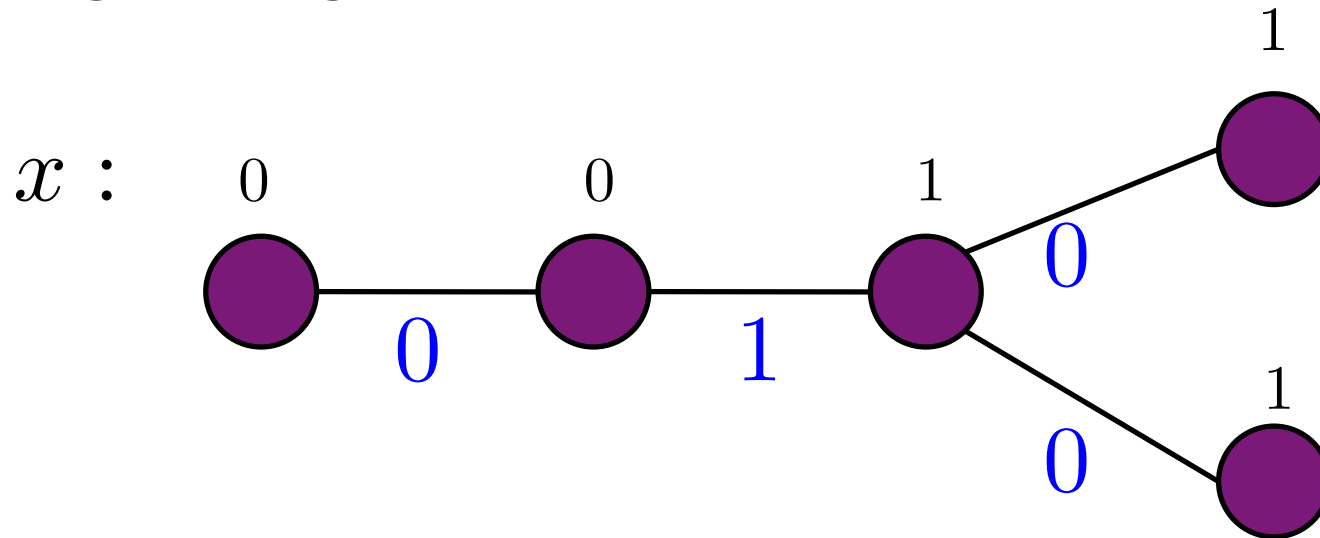
Laplacian Quadratic Form, examples

All edge-weights are 1



Laplacian Quadratic Form, examples

All edge-weights are 1

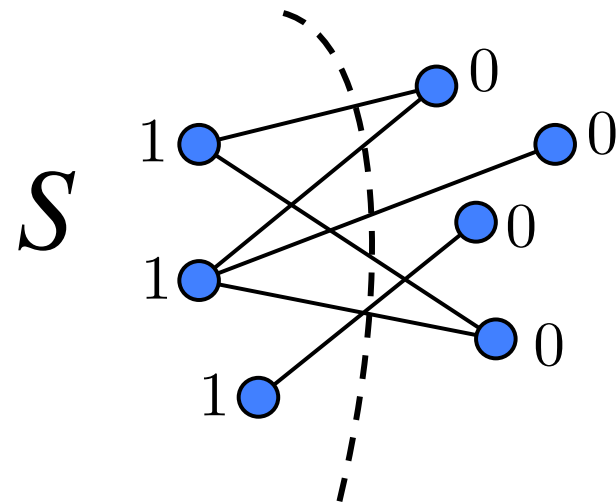


$$\begin{aligned} x^T L_G x &= \text{Sum of squares of} \\ &\text{differences across edges} \\ &= 1 \end{aligned}$$

Laplacian Quadratic Form, examples

When x is the characteristic vector of a set S ,
sum the weights of edges on the boundary of S

$$x^T L_G x = \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} w_{u,v}$$



Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2$$

$$L_G = D_G - A_G$$

Positive semi-definite

If connected, nullspace = Span(**1**)

Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2$$

$$L_G = \sum_{(u,v) \in E} w_{u,v} L_{u,v}$$

$$\begin{aligned} \text{E.g. } L_{1,2} &= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \end{aligned}$$

Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2$$

$$L_G = \sum_{(u,v) \in E} w_{u,v} L_{u,v}$$

$$= \sum_{(u,v) \in E} w_{u,v} (b_{u,v} b_{u,v}^T)$$

where $b_{u,v} = \delta_u - \delta_v$



Sum of outer products

Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$G \preceq H$$

If $L_H - L_G$ is positive semi-definite

Iff, for all $x \in R^V$

$$x^T L_G x \leq x^T L_H x$$

Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$G \preceq k \cdot H \quad (\text{multiply edge weights by } k)$$

If $k \cdot L_H - L_G$ is positive semi-definite

Iff, for all $x \in R^V$

$$x^T L_G x \leq k \cdot x^T L_H x$$

Approximation

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

H is an ϵ -approximation of G if

$$(1 + \epsilon)^{-1}G \preceq H \preceq (1 + \epsilon)G$$

That is, for all $x \in R^V$

$$\frac{1}{1 + \epsilon} \leq \frac{\sum_{(u,v) \in F} z_{u,v} (x(u) - x(v))^2}{\sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2} \leq 1 + \epsilon$$

Implications of Approximation

$$(1 + \epsilon)^{-1}G \preceq H \preceq (1 + \epsilon)G$$

L_H and L_G have similar eigenvalues

$$(1 + \epsilon)^{-1}x^T L_H^+ x \leq x^T L_G^+ x \leq (1 + \epsilon)x^T L_H^+ x$$

Effective resistances between vertices similar.

L_H is a good preconditioner for L_G

Spectral Sparsification [S-Teng]

For an input graph G with n vertices,

find a sparse graph H having $\tilde{O}(n)$ edges

so that H is an ϵ -approximation of G

Expanders Approximate Complete Graphs

Expanders:

d -regular graphs on n vertices

every set of vertices has large boundary

random walks mix quickly

weak expanders: eigenvalues bounded from 0

strong expanders: all eigenvalues near d

Example: Approximating a Complete Graph

For G the complete graph on n verts.

all non-zero eigs of L_G are n .

For $x \perp \mathbf{1}$, $\|x\| = 1$ $x^T L_G x = n$

Example: Approximating a Complete Graph

For G the complete graph on n verts.
all non-zero eigs of L_G are n .

$$\text{For } x \perp \mathbf{1}, \|x\| = 1 \quad x^T L_G x = n$$

For H a d -regular strong expander,
all non-zero eigs of L_H are close to d .

$$\text{For } x \perp \mathbf{1}, \|x\| = 1 \quad x^T L_H x \sim d$$

Example: Approximating a Complete Graph

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$$\text{For } x \perp \mathbf{1}, \|x\| = 1 \quad x^T L_H x \sim d$$

$\frac{n}{d}H$ is a good approximation of G

Best Approximations of Complete Graphs

Ramanujan Expanders

[Margulis, Lubotzky-Phillips-Sarnak]

$$d - 2\sqrt{d - 1} \leq \lambda(L_H) \leq d + 2\sqrt{d - 1}$$

Best Approximations of Complete Graphs

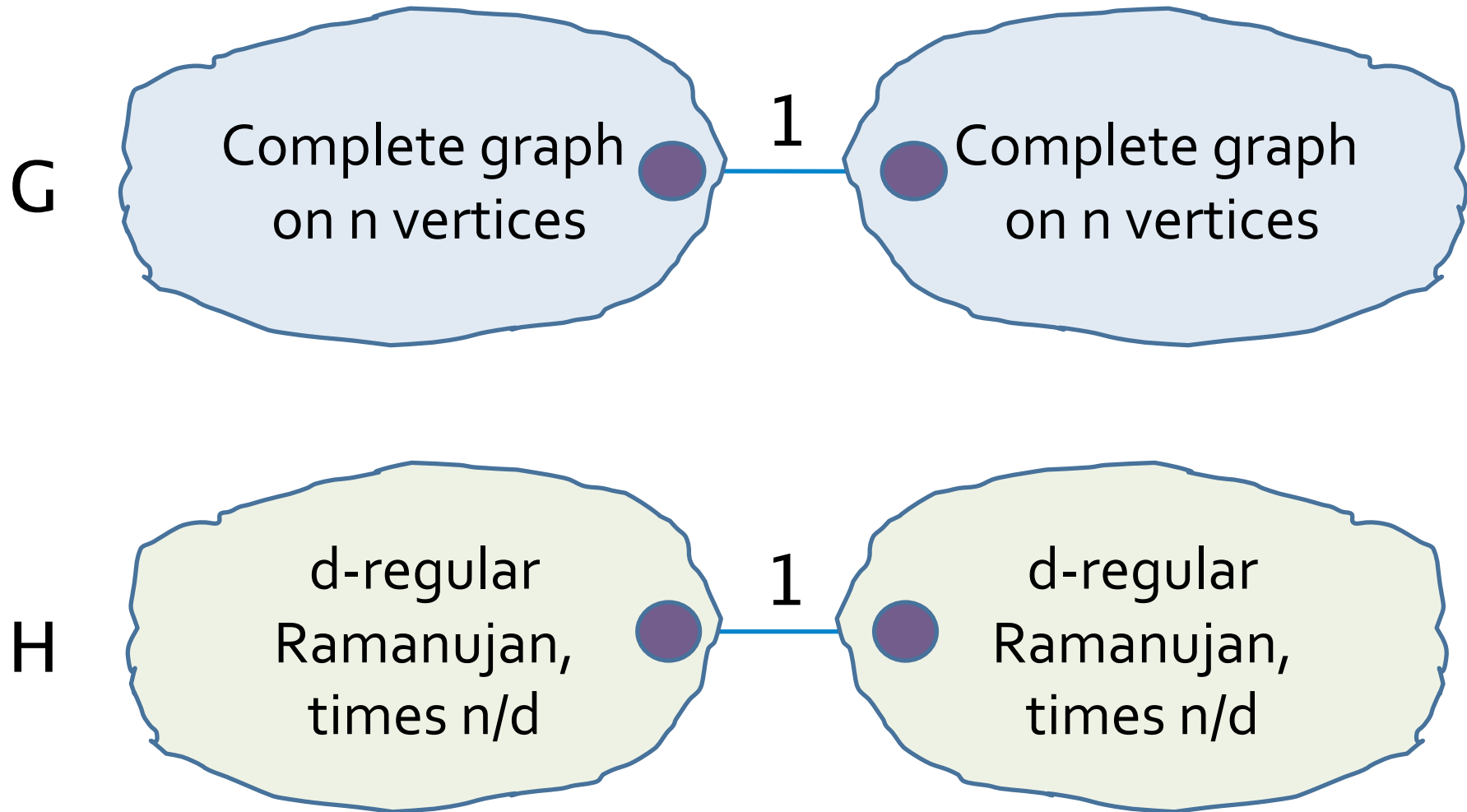
Ramanujan Expanders

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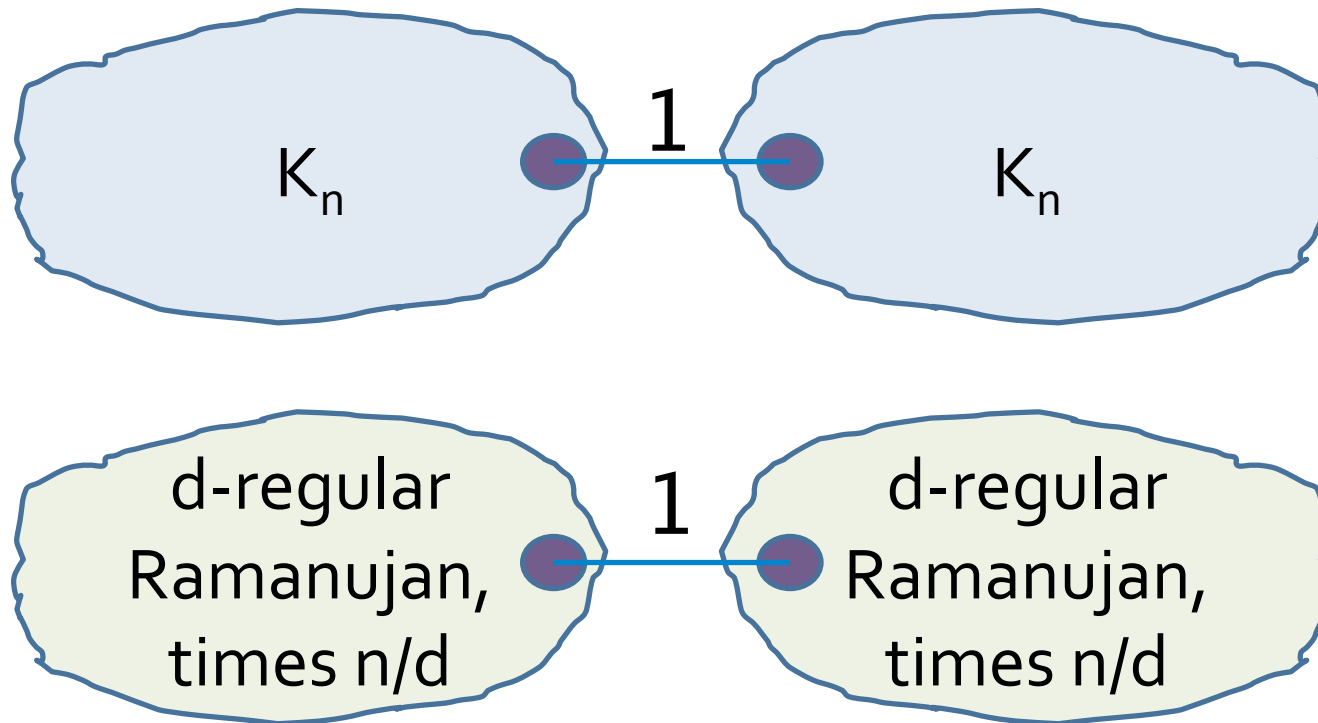
$$d - 2\sqrt{d-1} \leq \lambda(L_H) \leq d + 2\sqrt{d-1}$$

Can we approximate every graph this well?

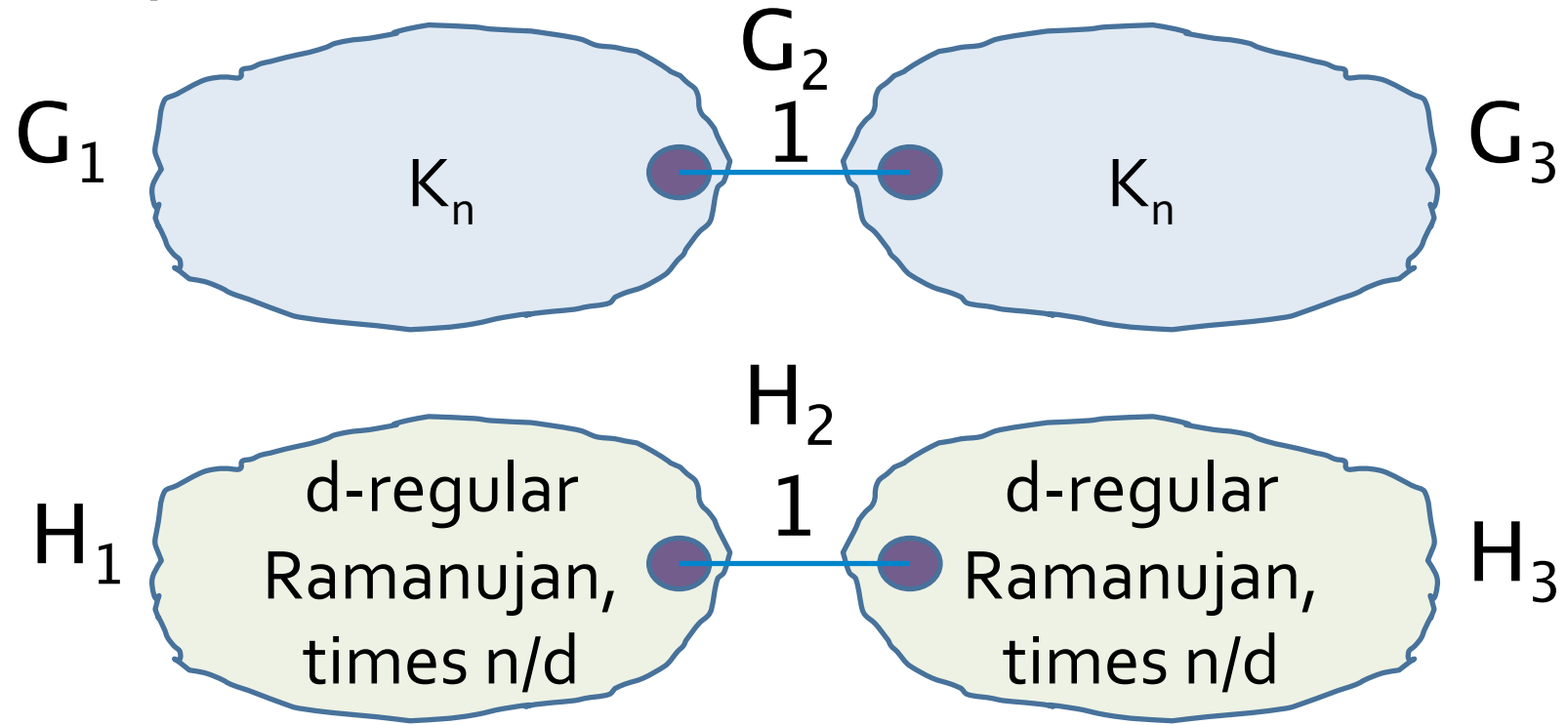
Example: Dumbbell



Example: Dumbbell



Example: Dumbbell



$$G = G_1 + G_2 + G_3$$

$$G_1 \preceq (1 + \epsilon)H_1$$

$$G \preceq (1 + \epsilon)H$$

$$H = H_1 + H_2 + H_3$$

$$G_2 = H_2$$

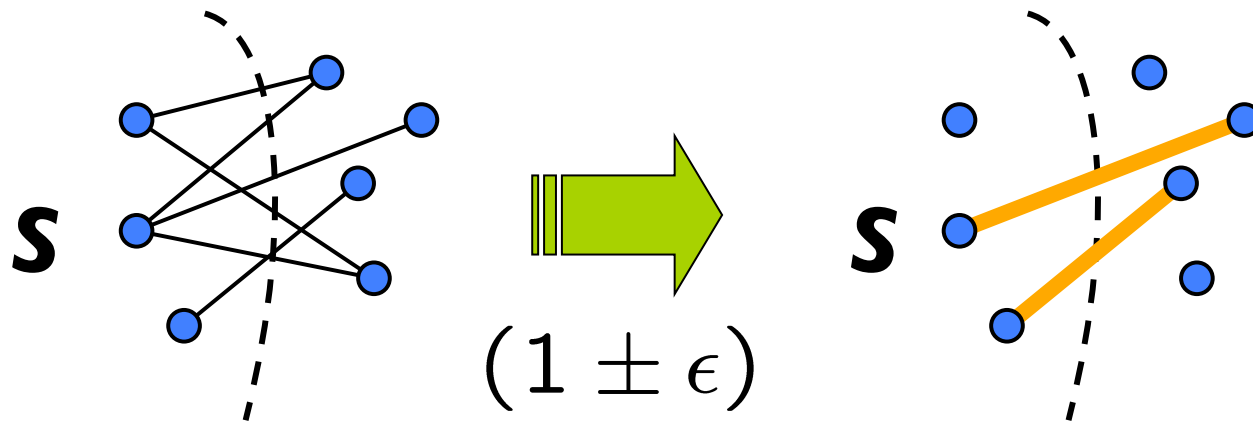
$$G_3 \preceq (1 + \epsilon)H_3$$

Cut-Sparsifiers [Benczur-Karger '96]

For every G , is an H with $O(n \log n / \epsilon^2)$ edges

for all $x \in \{0, 1\}^n$

$$(1 - \epsilon)x^T L_G x \leq x^T L_H x \leq (1 + \epsilon)x^T L_G x$$



Main Theorems

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

1. H is an ϵ -approximation of G
2. $|F| \leq 4|V|/\epsilon^2$ (Batson-S-Srivastava 09)
3. $F \subseteq E$

By careful random sampling, get

$$|F| \leq O(|V| \log |V| / \epsilon^2) \quad (\text{S-Srivastava 08})$$

Main Theorems (today)

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

1. H is a 2.6-approximation of G
2. $|F| \leq 6|V|$ (Batson-S-Srivastava 09)
3. $F \subseteq E$

By careful random sampling, get

$$|F| \leq O(|V| \log |V| / \epsilon^2) \quad (\text{S-Srivastava 08})$$

Why?

Solving linear equations in Laplacian Matrices
key part of nearly-linear time algorithm
use for learning on graphs, maxflow, PDEs, ...

Preserve Eigenvectors, Eigenvalues
and electrical properties

Generalize Expanders

Certifiable cut-sparsifiers

Other matrix approximations

Random Sampling [Achlioptas-McSherry '01,
Füredi-Komlós '81, Vu '07]

Low-rank approximations

[Frieze-Kannan-Vempala '98, Rudelson-Vershynin '07]

Work for all matrices, but have additive error

Reduction to Matrix Approximation

$$1 \leq \frac{x^T L_H x}{x^T L_G x} \leq 13 \quad \forall x \perp \mathbf{1}$$

Setting $x = L_G^{-1/2} z$

Becomes
$$1 \leq \frac{z^T L_G^{-1/2} L_H L_G^{-1/2} z}{z^T z} \leq 13$$

$$\forall z \perp \mathbf{1}$$

Reduction to Matrix Approximation

Suffices to show

$$1 \leq \frac{z^T L_G^{-1/2} L_H L_G^{-1/2} z}{z^T z} \leq 13 \quad \forall z \perp \mathbf{1}$$

$$1 \leq \lambda \left(L_G^{-1/2} L_H L_G^{-1/2} \right) \leq 13$$

Reduction to Matrix Approximation

$$1 \leq \lambda \left(L_G^{-1/2} L_H L_G^{-1/2} \right) \leq 13$$

Write $L_H = \sum s_e (b_e b_e^T)$

*weight of edge e
in graph H*

*Laplacian of edge e ,
as outer-product of vectors*

Need $1 \leq \lambda \left(\sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} \right) \leq 13$

At most $6n$ of the s_e non-zero

Reduction to Matrix Approximation

Need $1 \leq \lambda \left(\sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} \right) \leq 13$

At most $6n$ of the s_e non-zero

Reduction to Matrix Approximation

Need $1 \leq \lambda\left(\sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2}\right) \leq 13$

At most $6n$ of the s_e non-zero

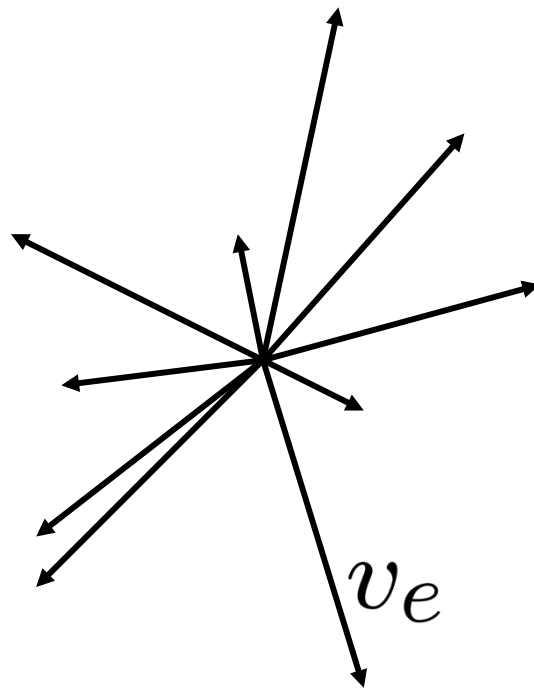
$$1 \leq \lambda\left(\sum_e s_e v_e v_e^T\right) \leq 13$$

Where $v_e = L_G^{-1/2} b_e$

$$\sum_e v_e v_e^T = \sum_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} = L_G^{-1/2} L_G L_G^{-1/2} = id$$

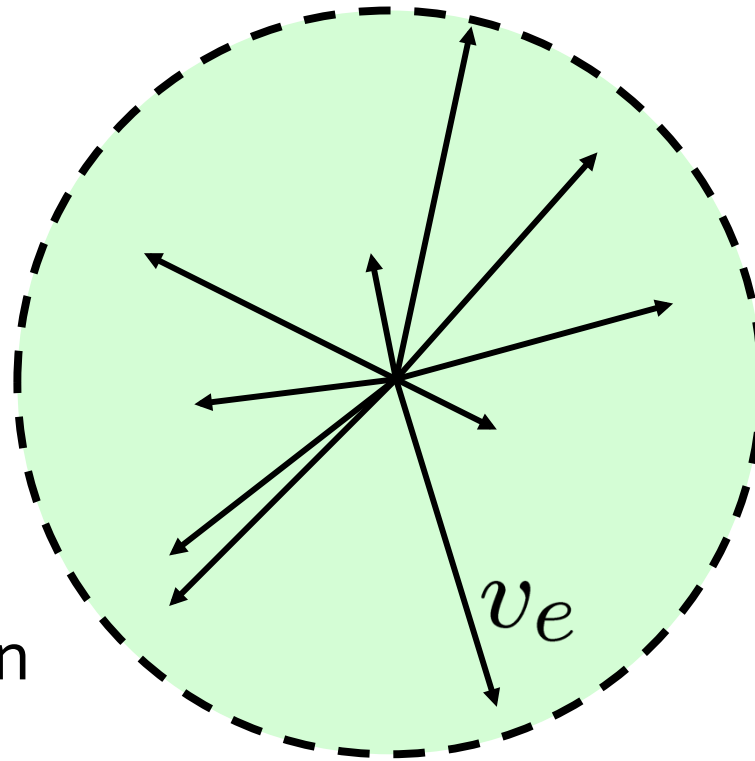
A closer look at v_e

$$v_e = L_G^{-1/2} b_e.$$



**m vectors
in \mathbf{R}^{n-1}**

A closer look at v_e

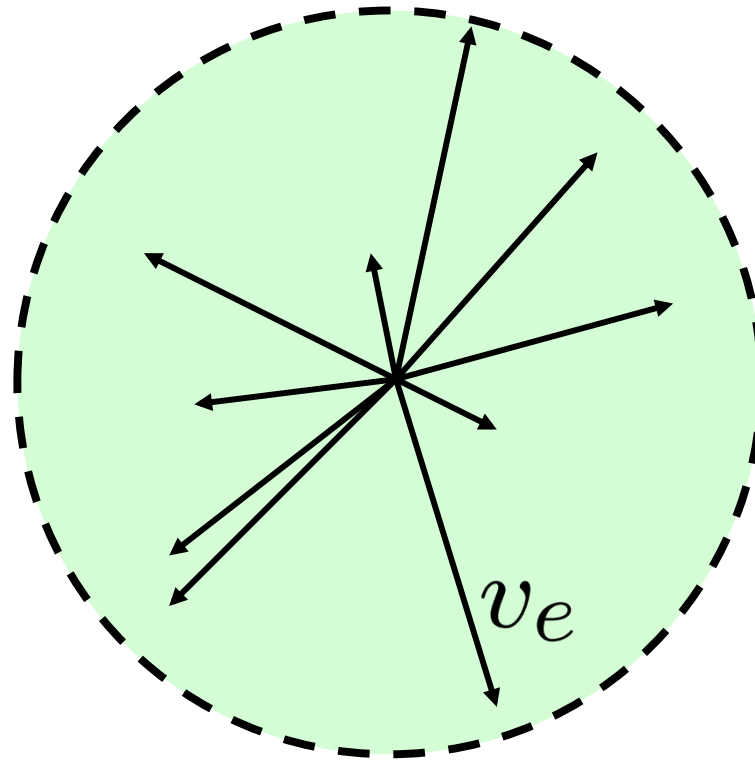


**m vectors
in R^{n-1}**

“decomposition
of identity”

$$\sum_e v_e v_e^T = L_G^{-1/2} \left(\sum_e b_e b_e^T \right) L_G^{-1/2} = I$$

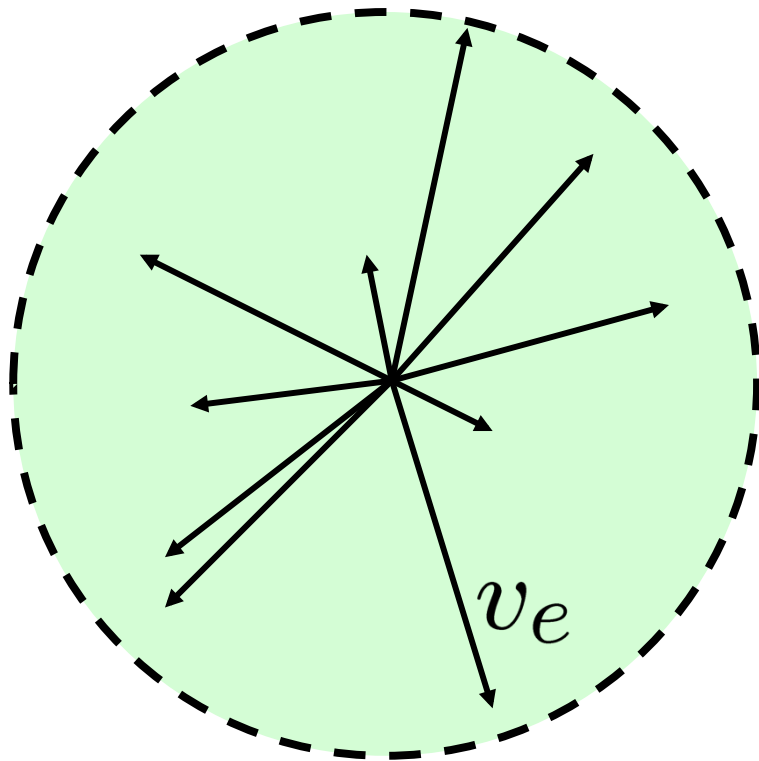
A closer look at v_e



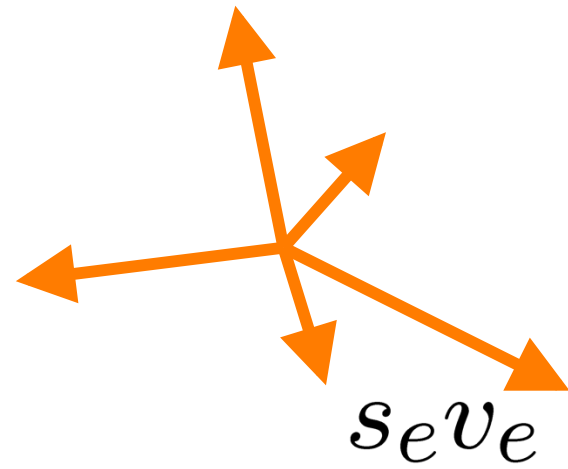
**m vectors
in R^{n-1}**

$$\forall u \quad \sum_e \langle u, v_e \rangle^2 = 1$$

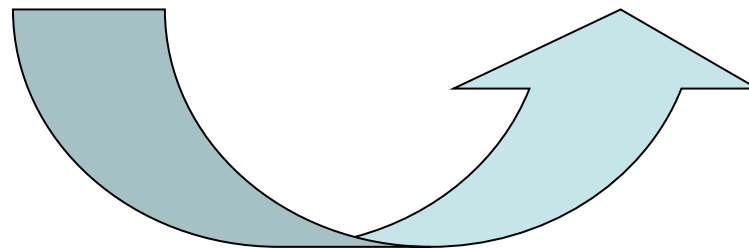
Choosing a Subgraph



G

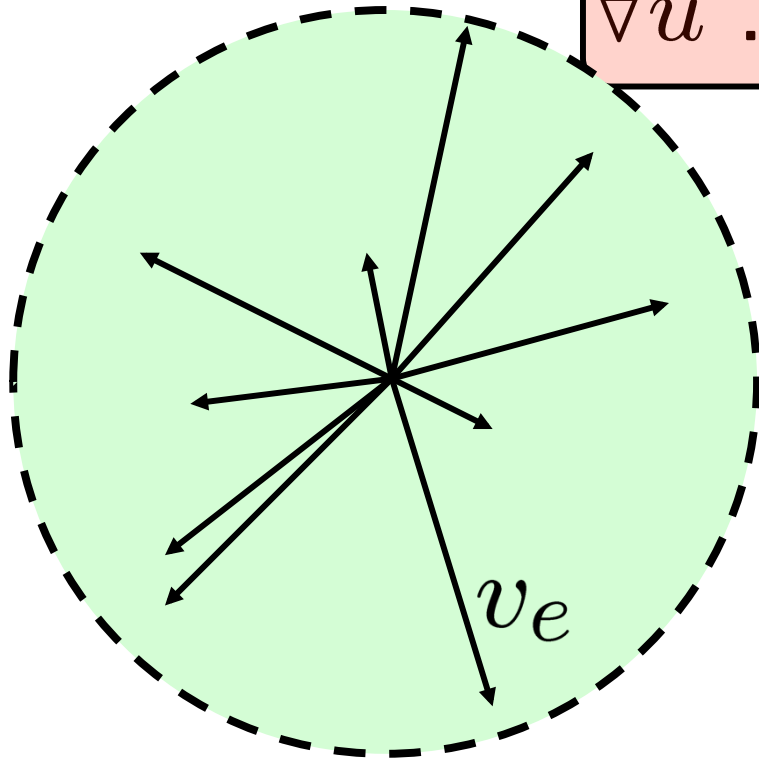


H

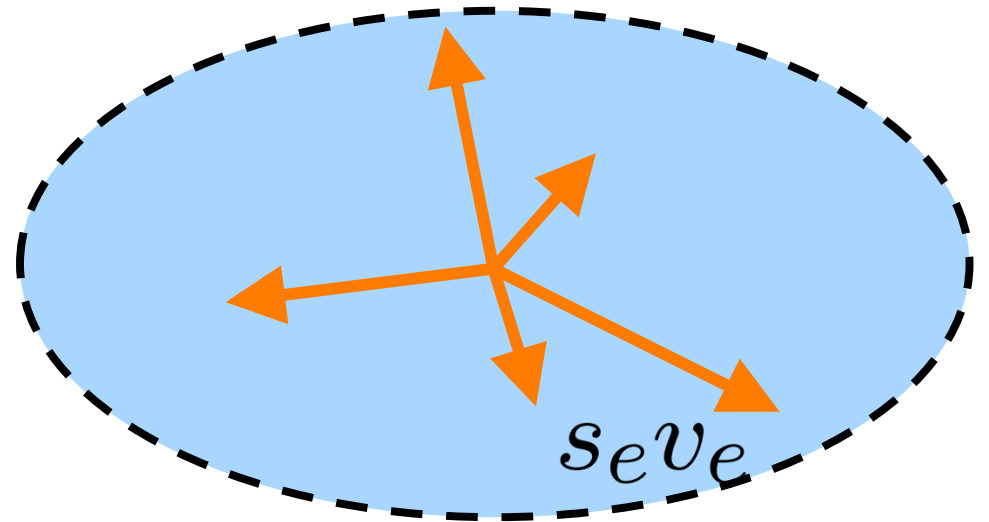


New Goal

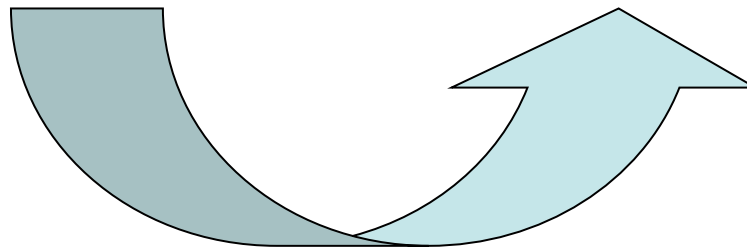
$$\forall u : 1 \leq \sum_e s_e \langle u, v_e \rangle^2 \leq 13$$



G



H



Existence theorem

If

$$\sum_e v_e v_e^T = I_n$$

then there are scalars $s_e \geq 0$ with

$$1 \leq \lambda\left(\sum_e s_e v_e v_e^T\right) \leq 13$$

and $|\{s_e \neq 0\}| \leq 6n$.

Existence theorem

If

$$\sum_e v_e v_e^T = I_n$$

then there are scalars $s_e \geq 0$ with

$$1 \leq \lambda\left(\sum_e s_e v_e v_e^T\right) \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$

and $|\{s_e \neq 0\}| \leq dn$

Efficient sampling theorem

If
$$\sum_e v_e v_e^T = I_n$$

Then there are scalars $s_e \geq 0$

$$1 \leq \lambda(\sum_e s_e v_e v_e^T) \leq 1 + \epsilon$$

$$|\{s_e \neq 0\}| \leq O(n \log n / \epsilon^2)$$

Proof: sample with prob proportional to $\|v_e\|^2$
(effective resistance between endpoints of e)
Rescale. Apply Rudelson's concentration theorem.
Nearly-linear time.

Existence theorem

If

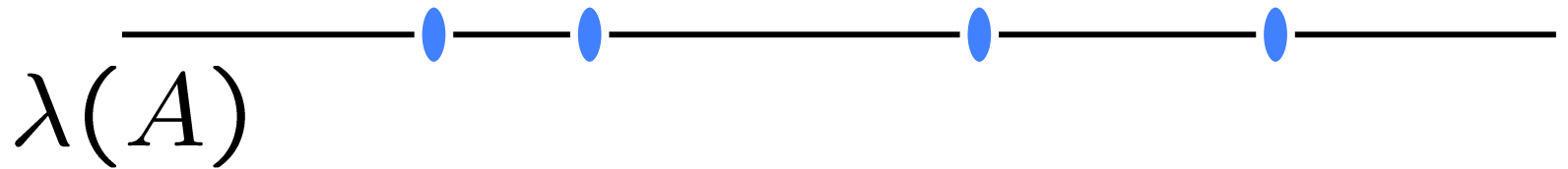
$$\sum_e v_e v_e^T = I_n$$

then there are scalars $s_e \geq 0$ with

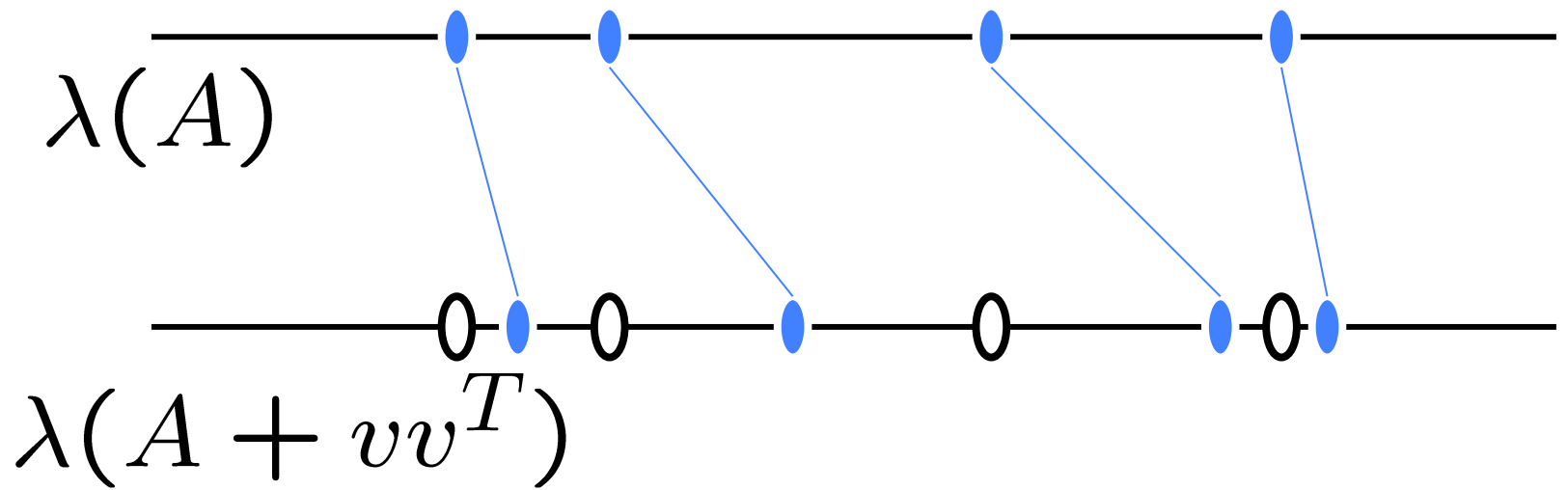
$$1 \leq \lambda\left(\sum_e s_e v_e v_e^T\right) \leq 13$$

and $|\{s_e \neq 0\}| \leq 6n$.

What happens when we add a vector?



Interlacing



More precisely

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

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$$p_A(x) = \det(xI - A)$$

Rank-one update:

$$p_{A+vv^T} = \left(1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x} \right) p_A$$

Where $Au_i = \lambda_i u_i$

More precisely

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

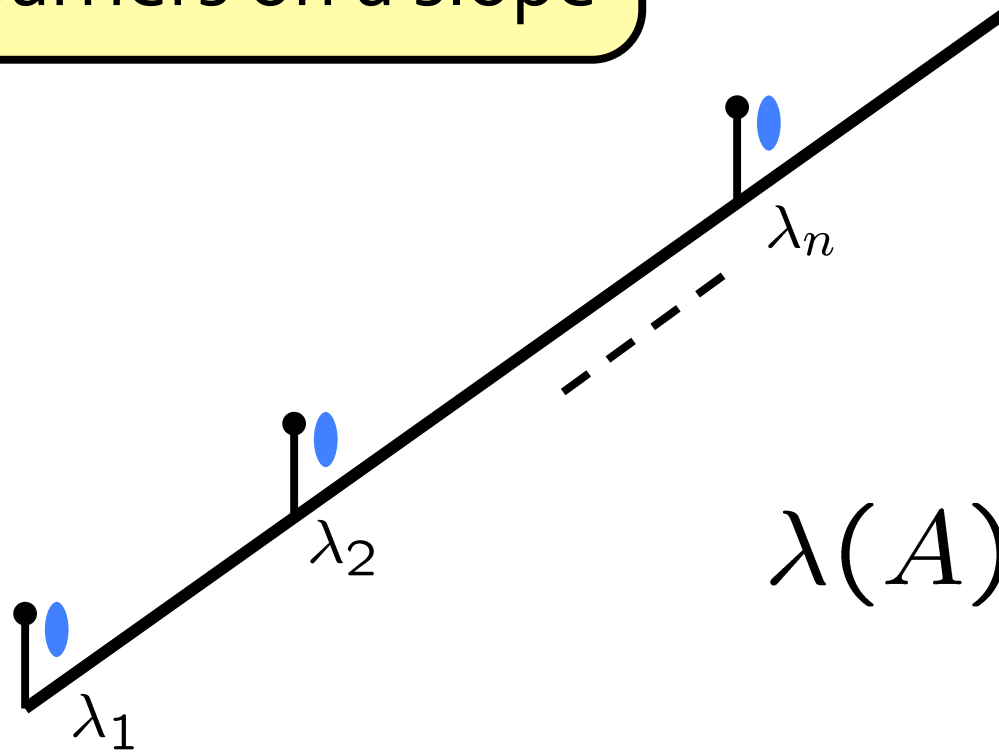
Rank-one update:

$$p_{A+vv^T} = \left(1 + \sum_i \frac{\langle v, u_i \rangle^2}{\lambda_i - x} \right) p_A$$

$\lambda(A + vv^T)$ are zeros of 

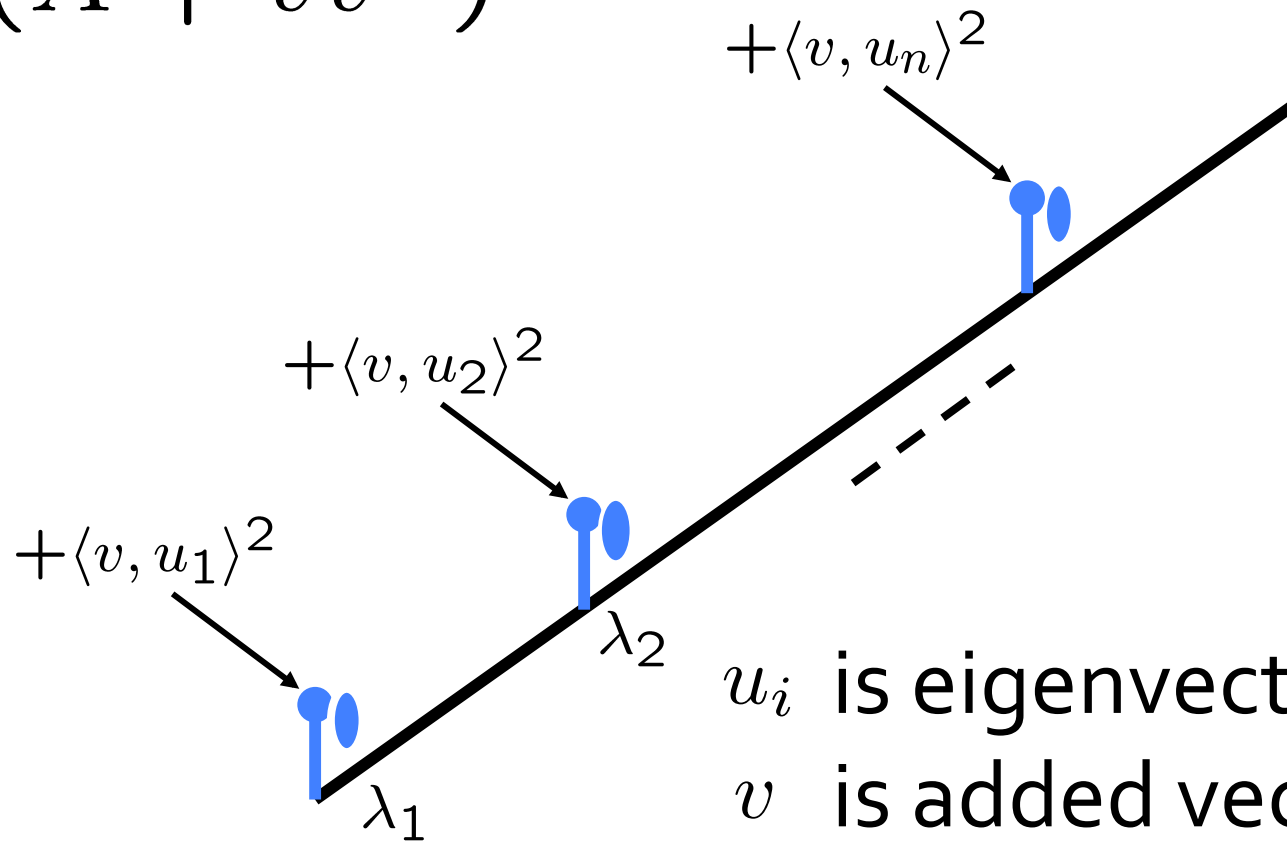
Physical model of interlacing

λ_i = positive unit charges
resting at barriers on a slope



Physical model of interlacing

$$\lambda(A + vv^T)$$



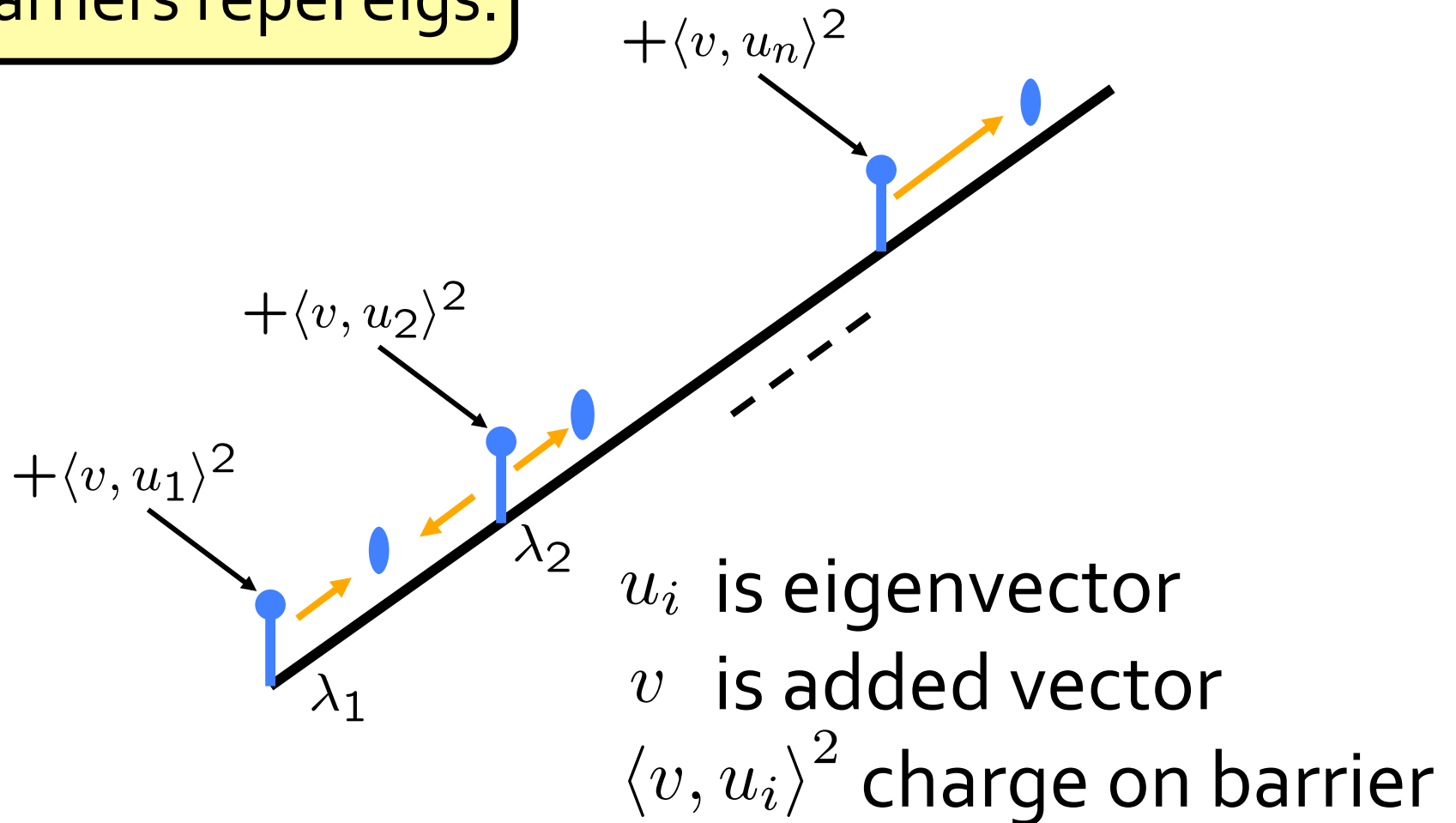
u_i is eigenvector

v is added vector

$\langle v, u_i \rangle^2$ charge on barrier

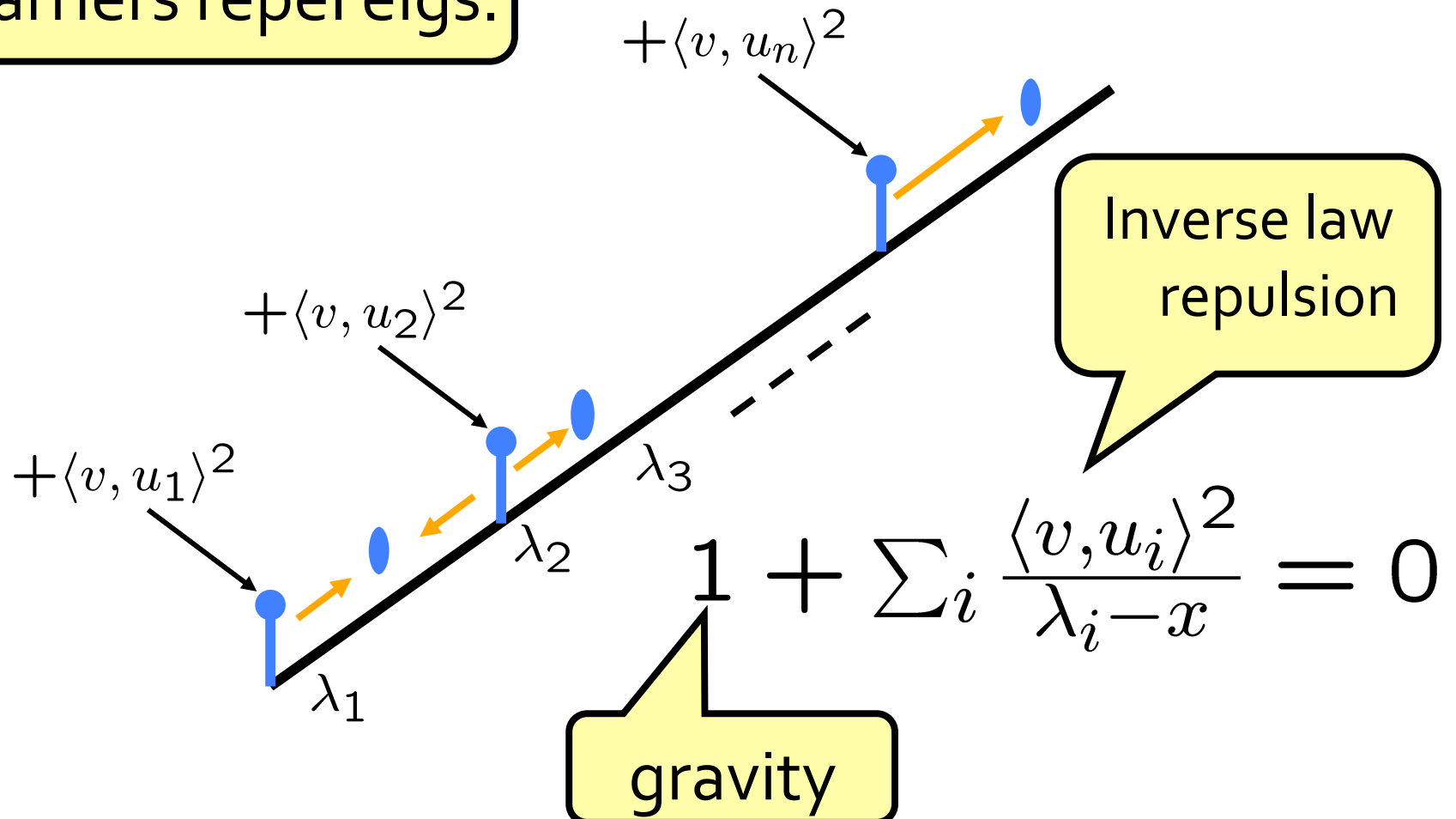
Physical model of interlacing

Barriers repel eigs.

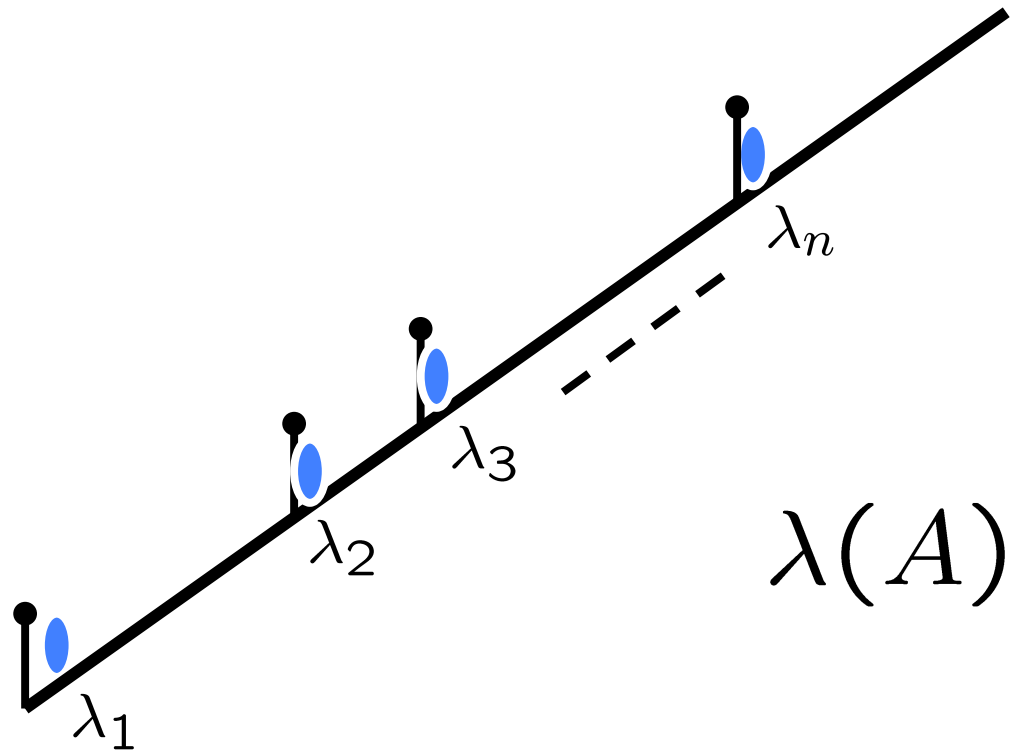


Physical model of interlacing

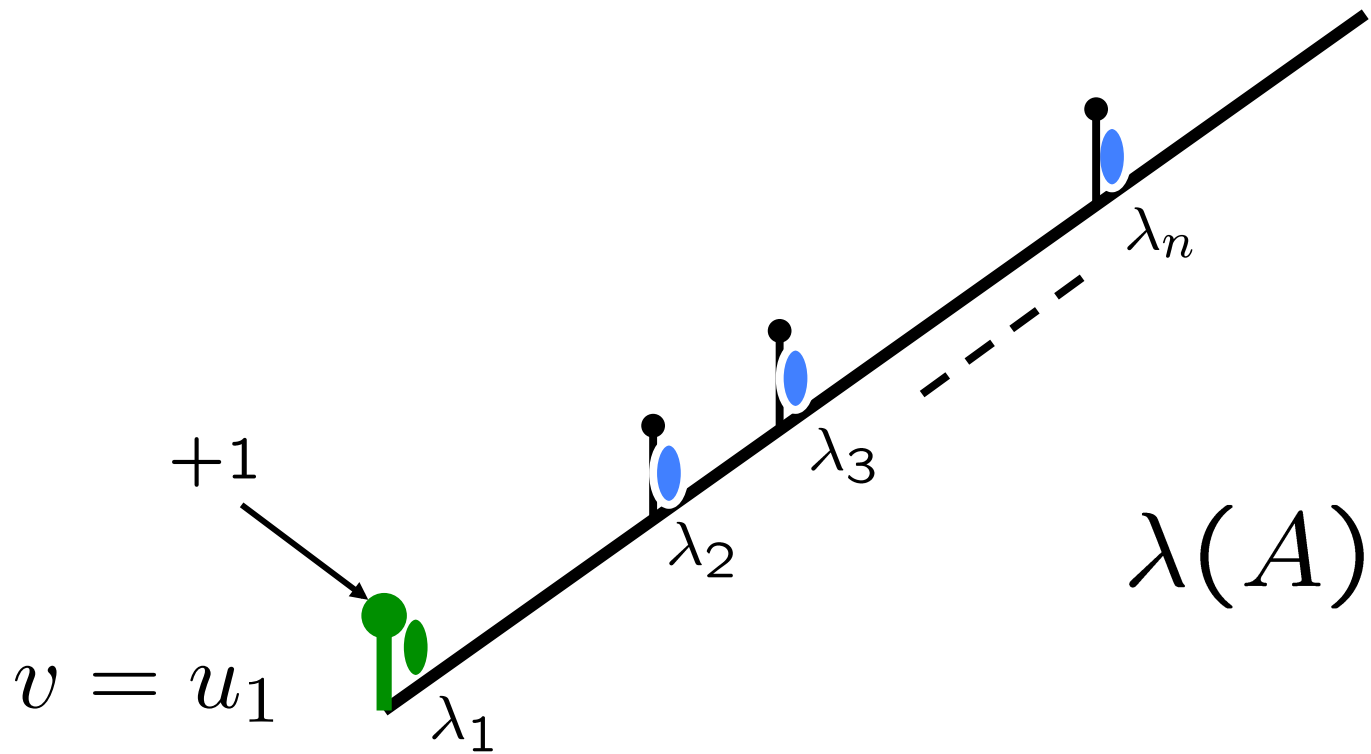
Barriers repel eigs.



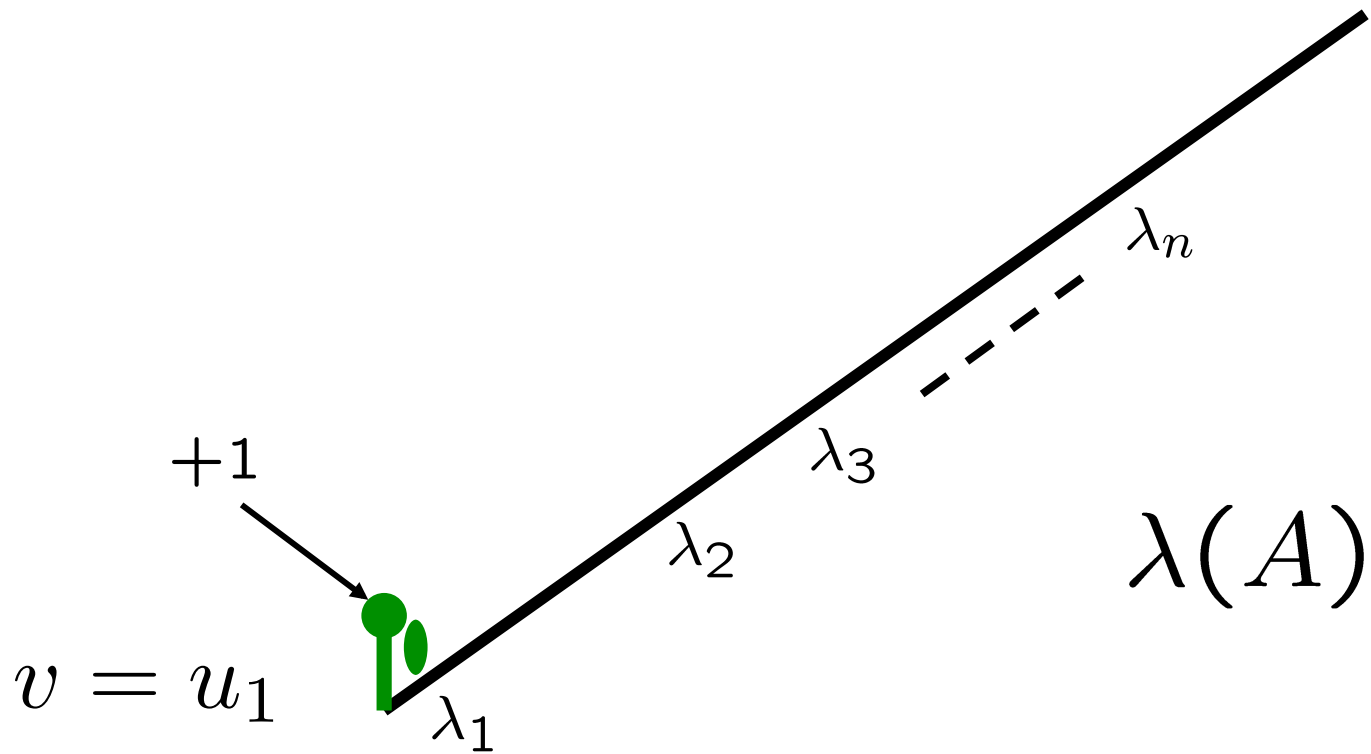
Examples



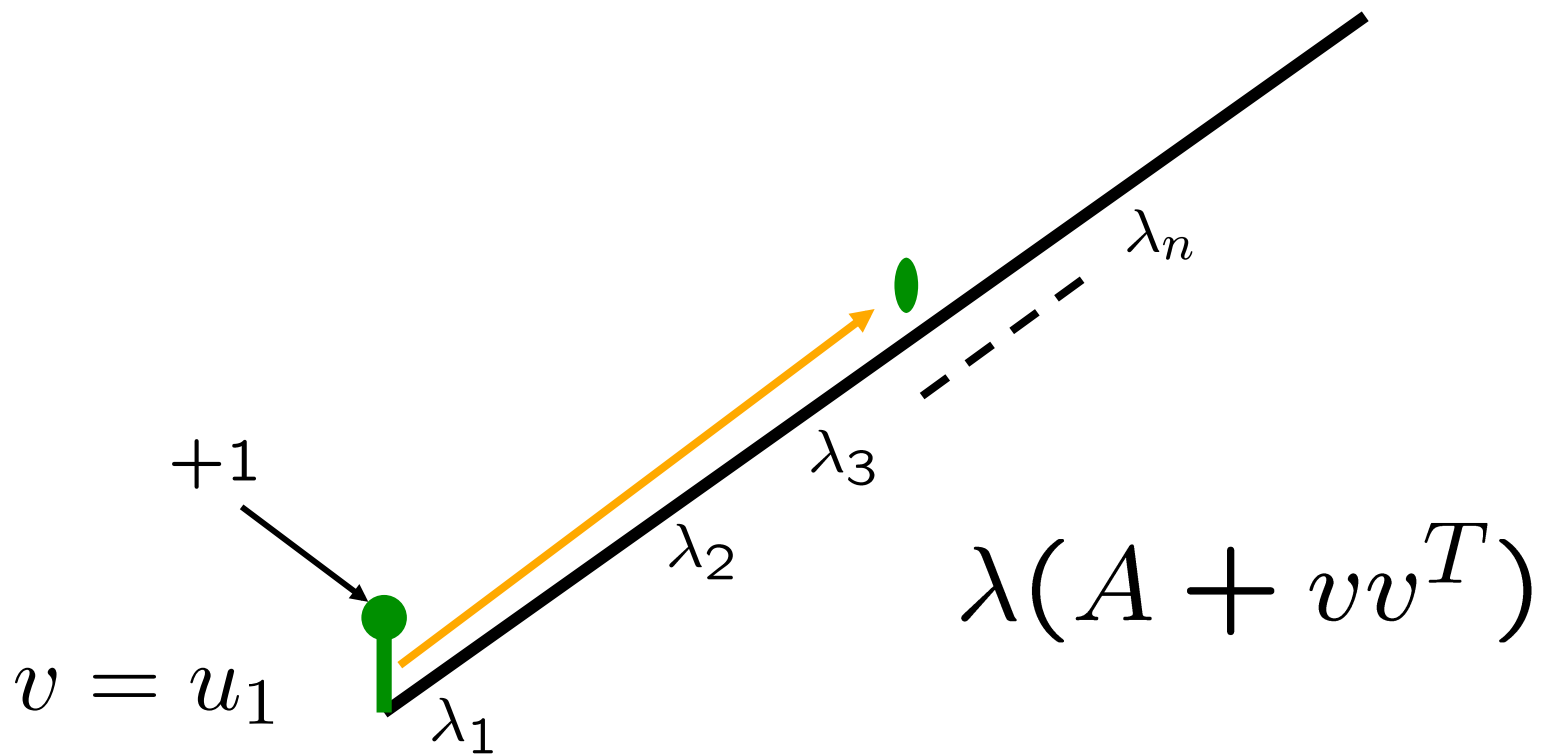
Ex1: All weight on u_1



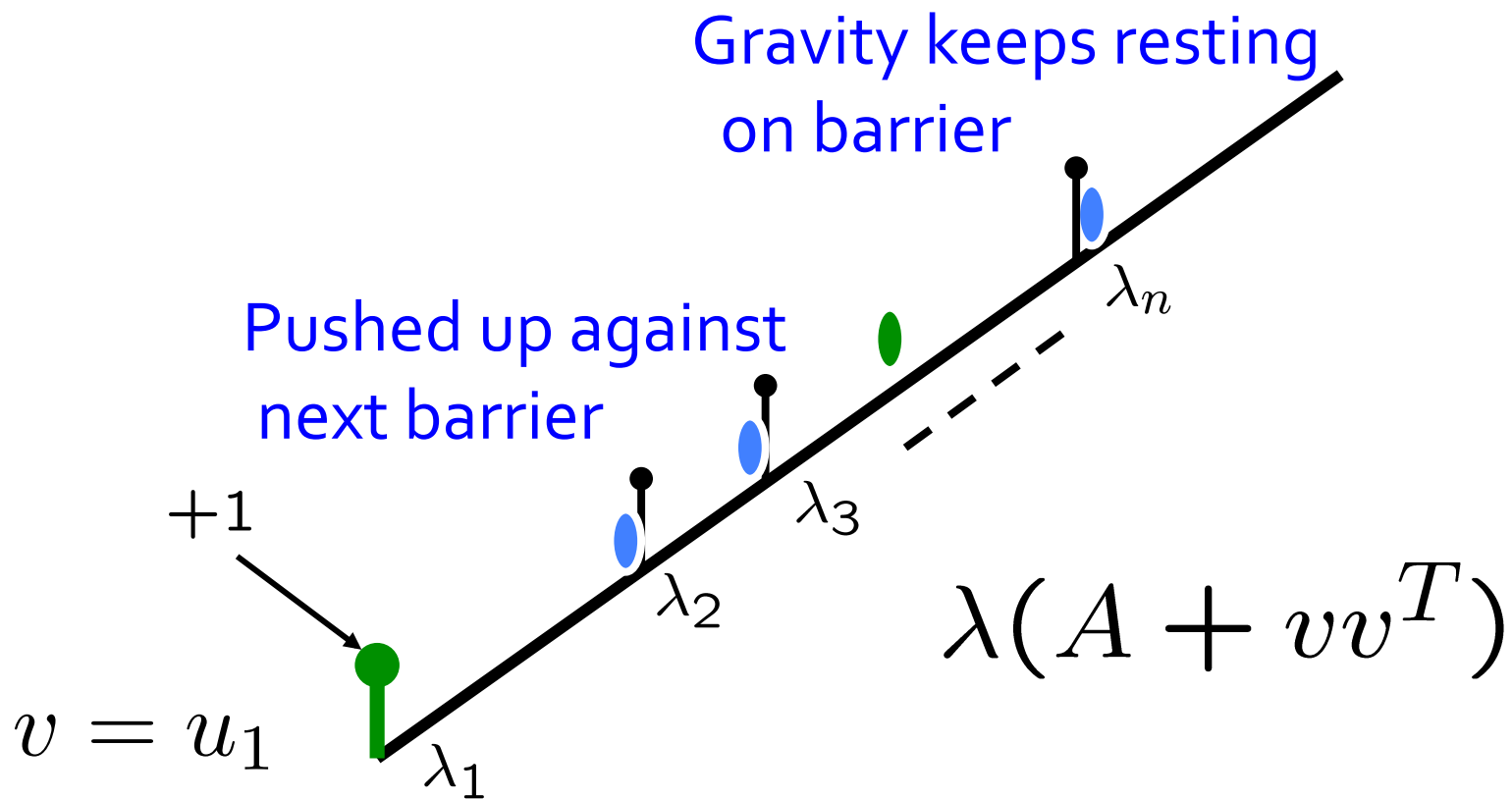
Ex1: All weight on u_1



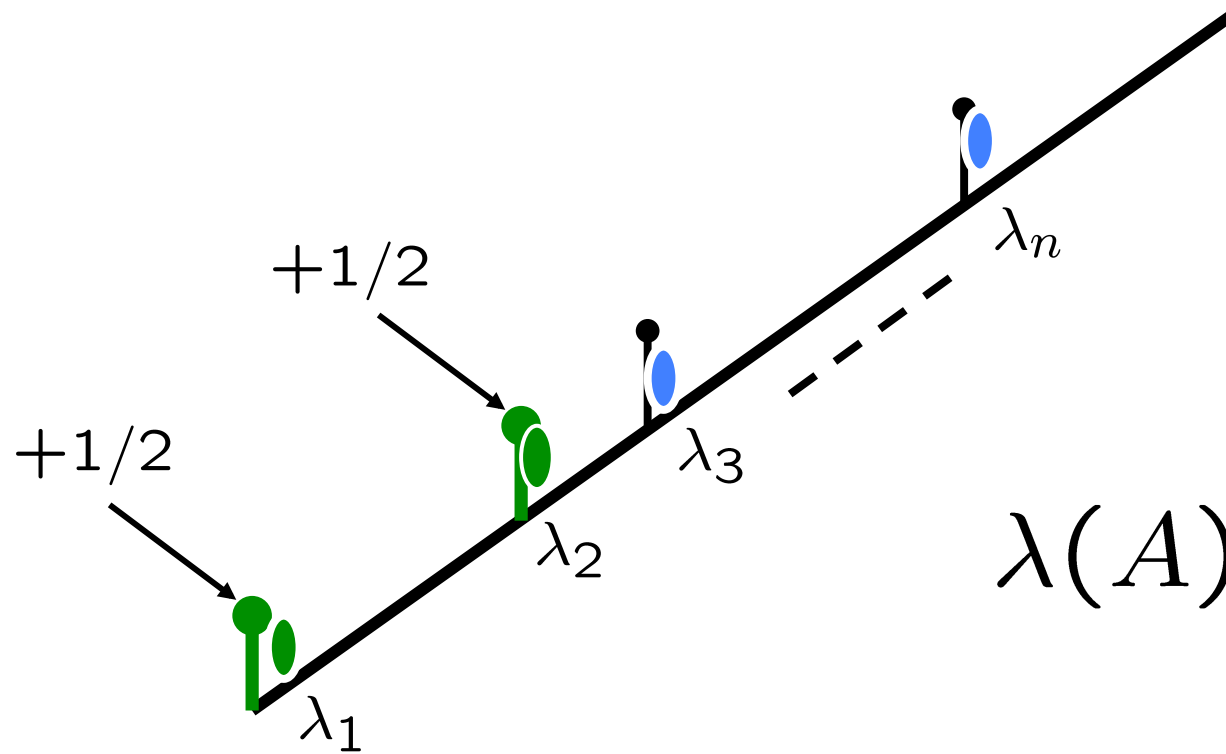
Ex1: All weight on u_1



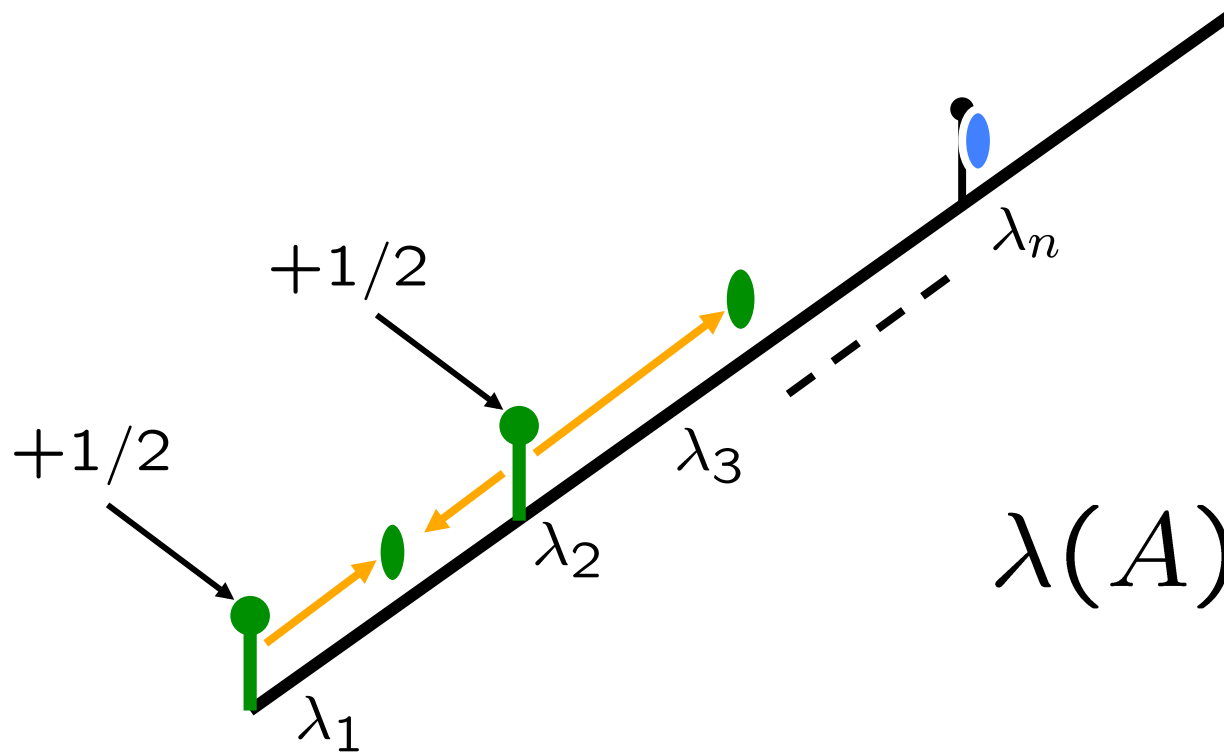
Ex1: All weight on u_1



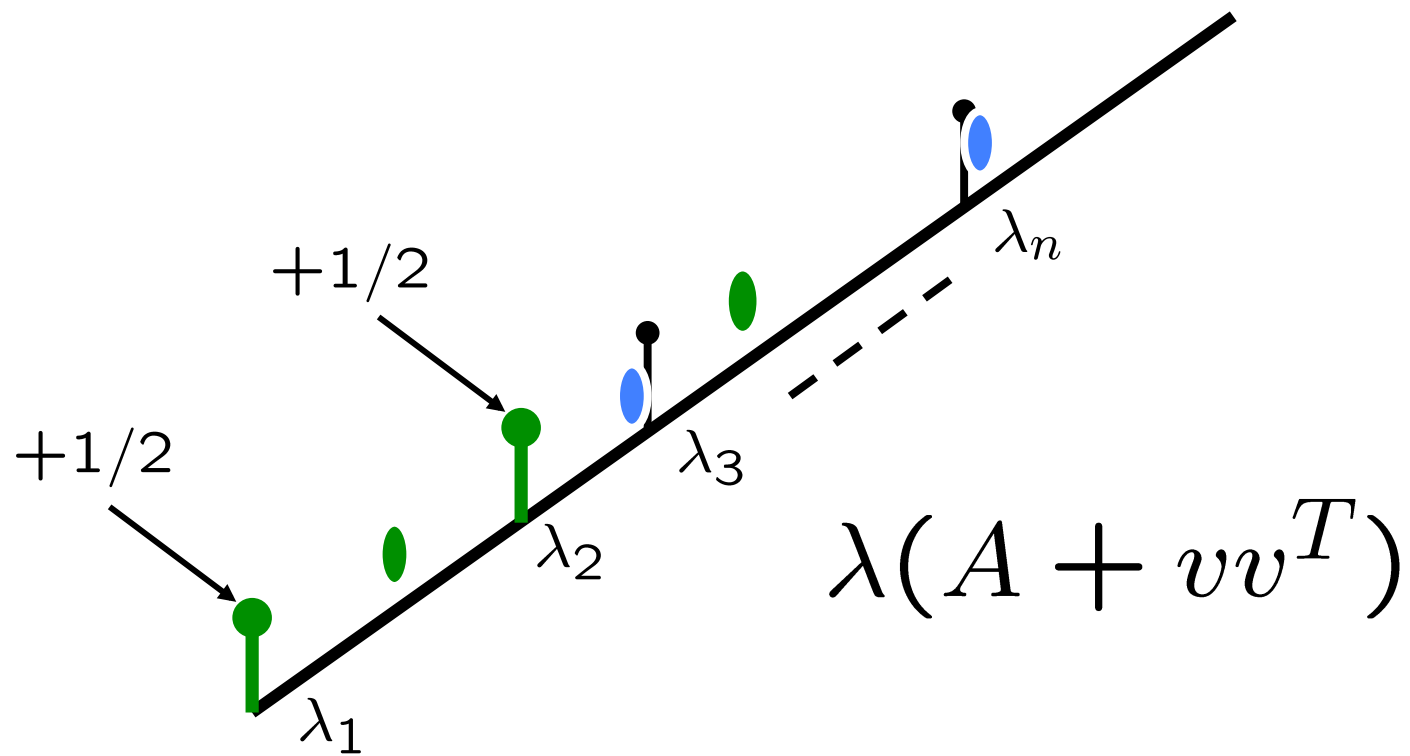
Ex2: Equal weight on u_1, u_2



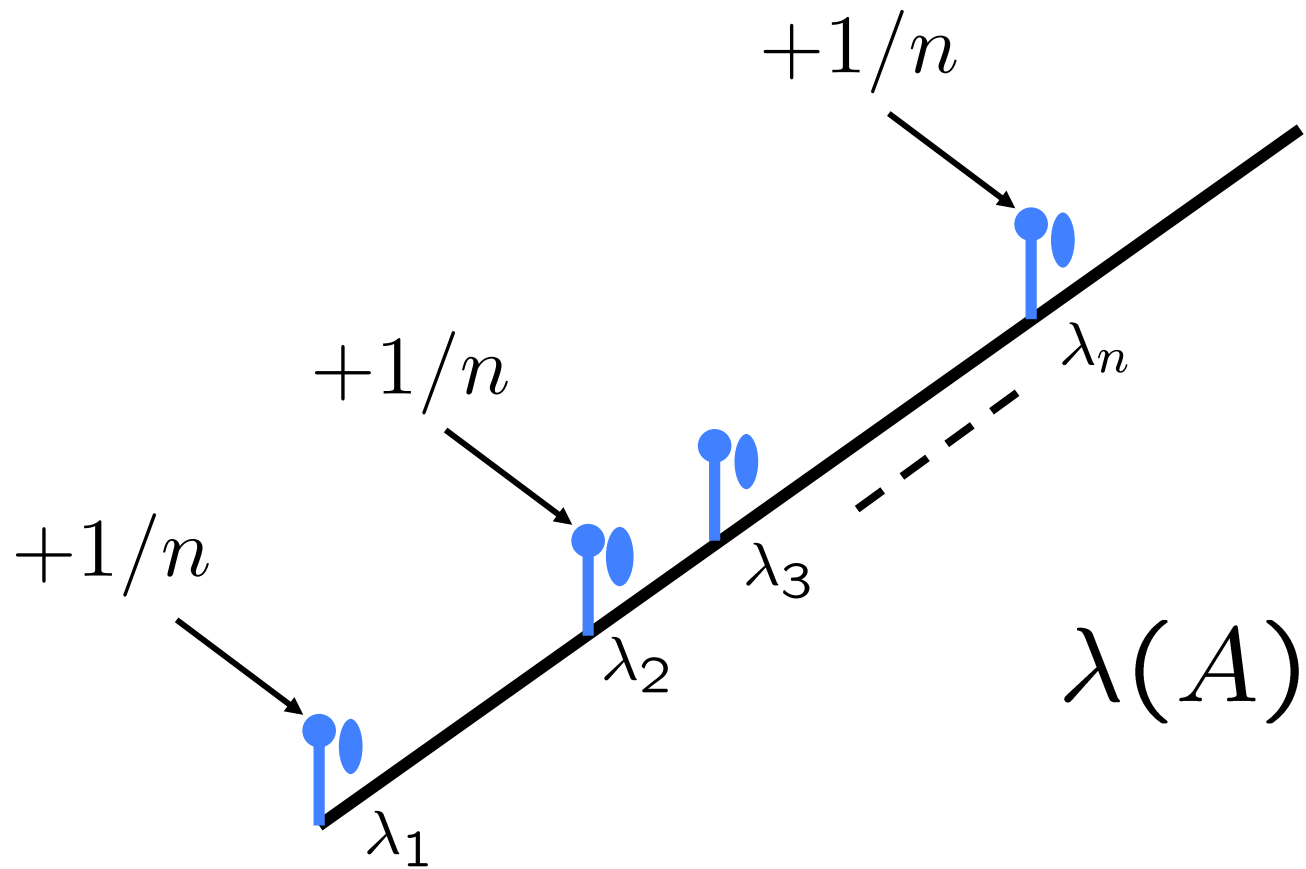
Ex2: Equal weight on u_1, u_2



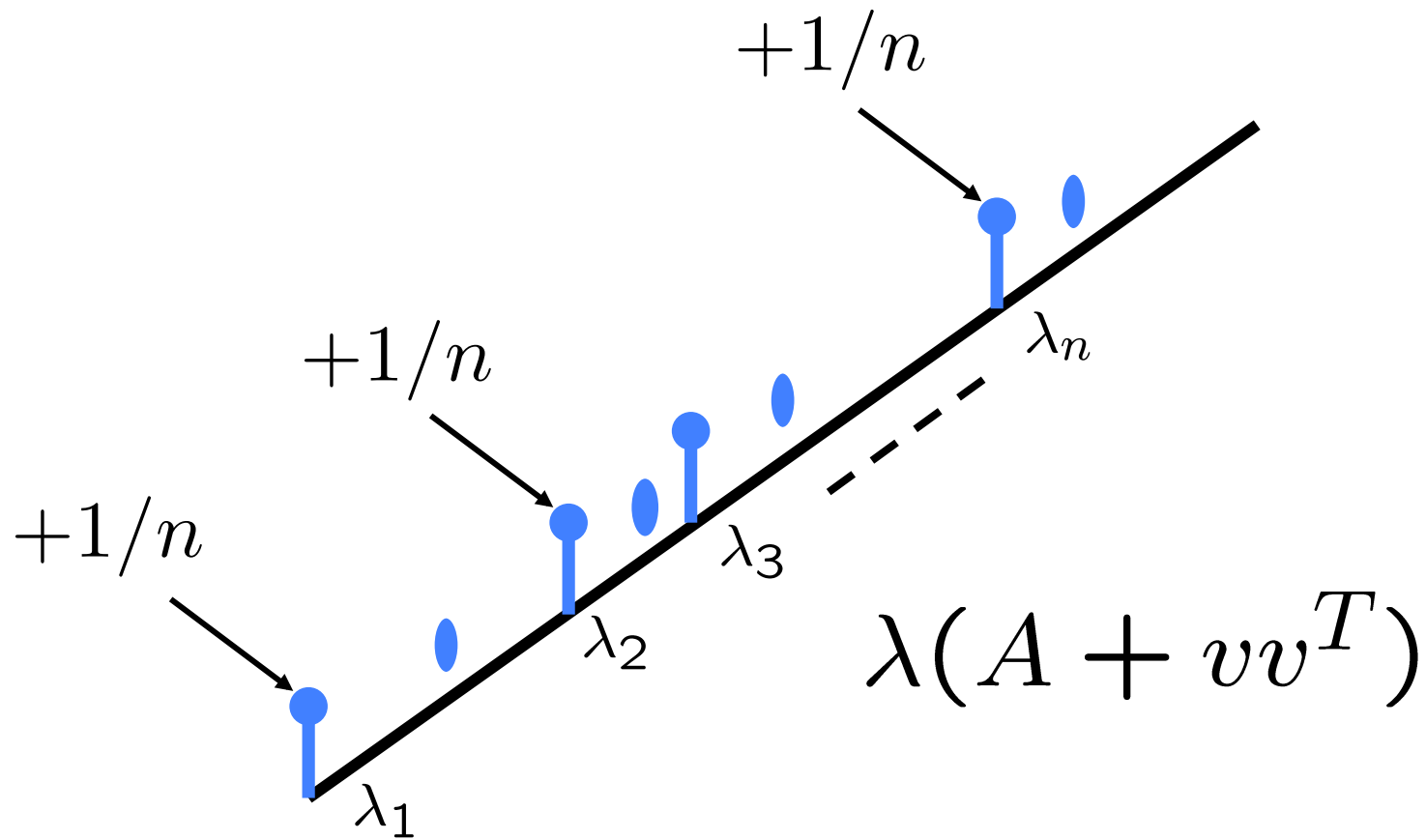
Ex2: Equal weight on u_1, u_2



Ex3: Equal weight on all u_1, u_2, \dots, u_n



Ex3: Equal weight on all u_1, u_2, \dots, u_n



Adding a random v_e

Because v_e are decomposition of identity,

$$\mathbf{E}_e \left[\langle v_e, u \rangle^2 \right] = 1/m$$

$$\begin{aligned} \mathbf{E}_e \left[P_{A+v_e v_e^T} \right] &= \left(1 + \frac{1}{m} \sum_i \frac{1}{\lambda_i - x} \right) P_A \\ &= P_A - \frac{1}{m} \frac{d}{dx} P_A \end{aligned}$$

Ideal proof




$$A^{(0)} = 0$$

$$p^{(0)} = x^n$$

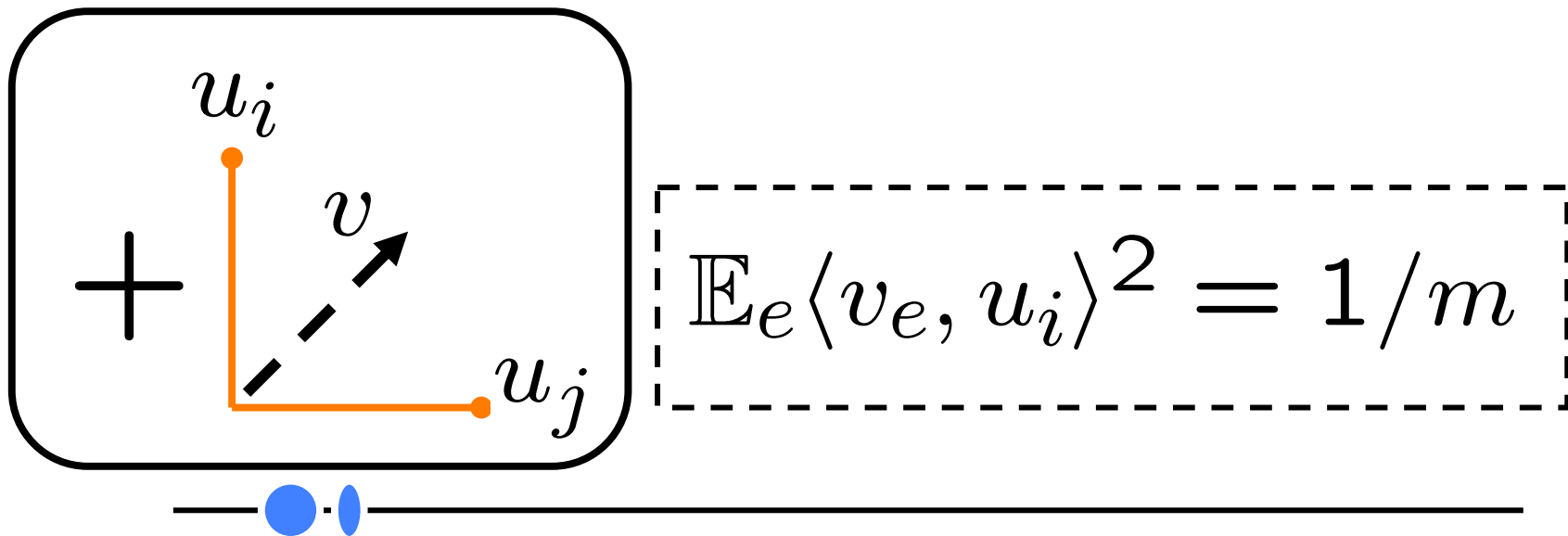
Ideal proof

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$


$$A^{(0)} = 0$$

$$p^{(0)} = x^n$$

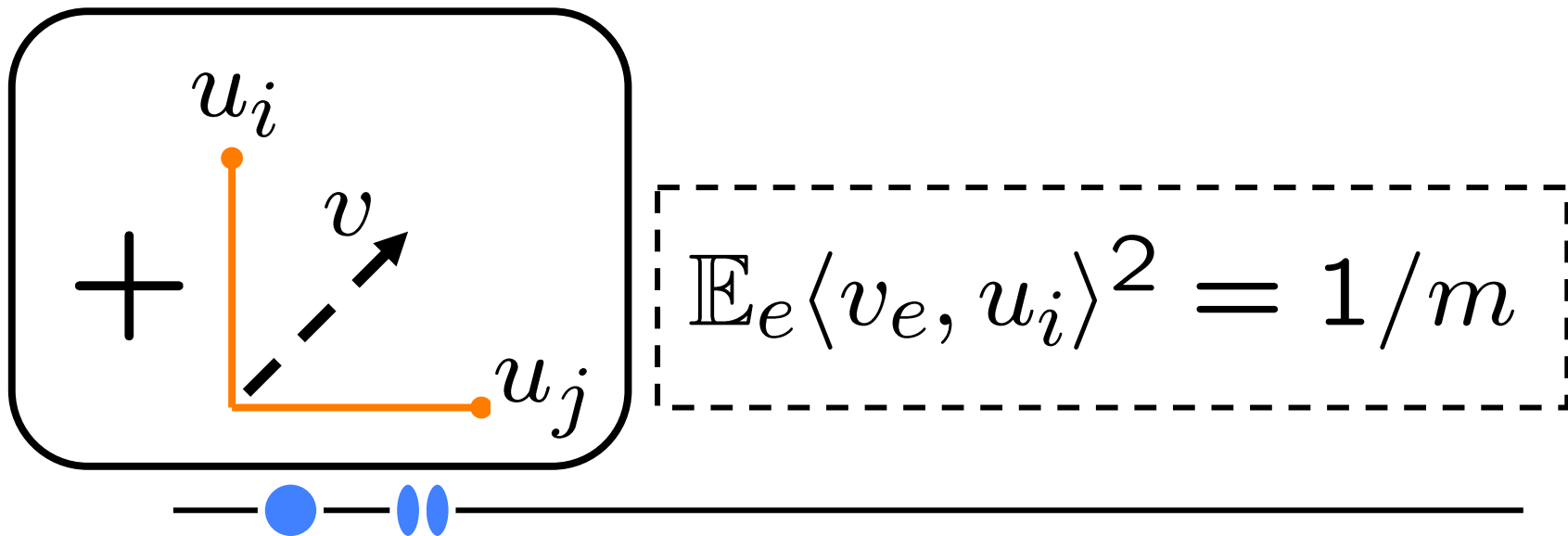
Ideal proof



$$A^{(1)} = 0 + vv^T$$

$$p^{(1)} = x^n - \frac{1}{m} \frac{d}{dx} x^n$$

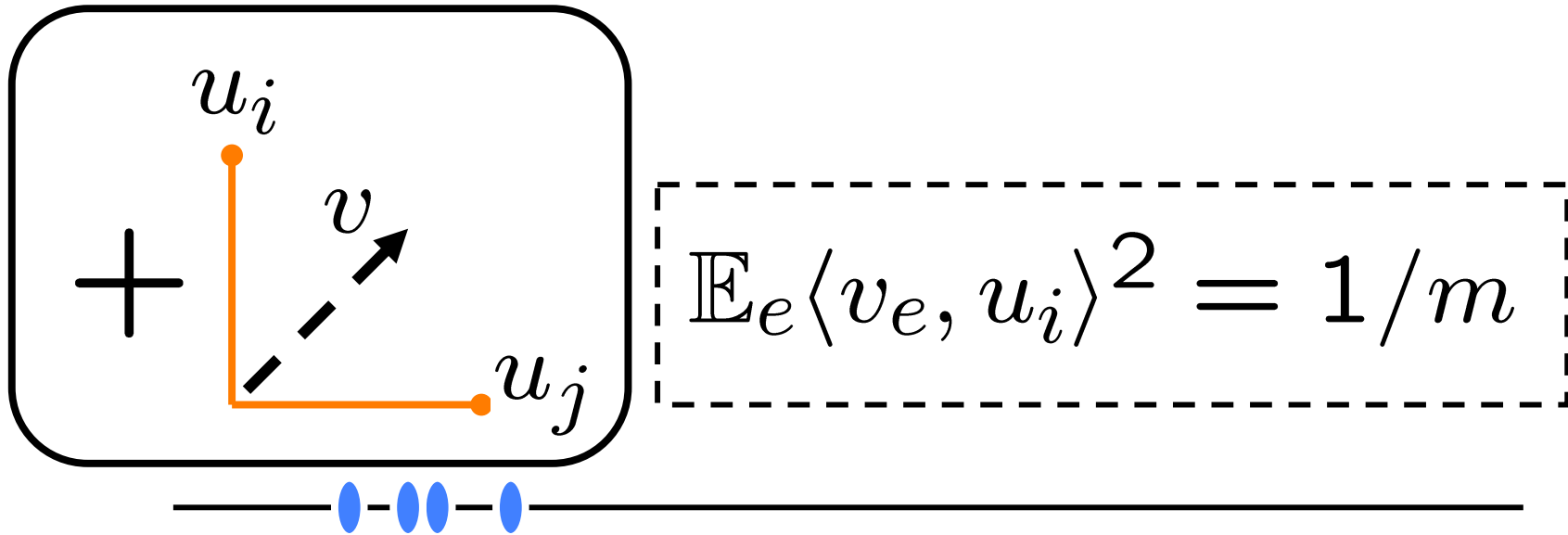
Ideal proof



$$A^{(2)} = A^{(1)} + vv^T$$

$$p^{(2)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^2 x^n$$

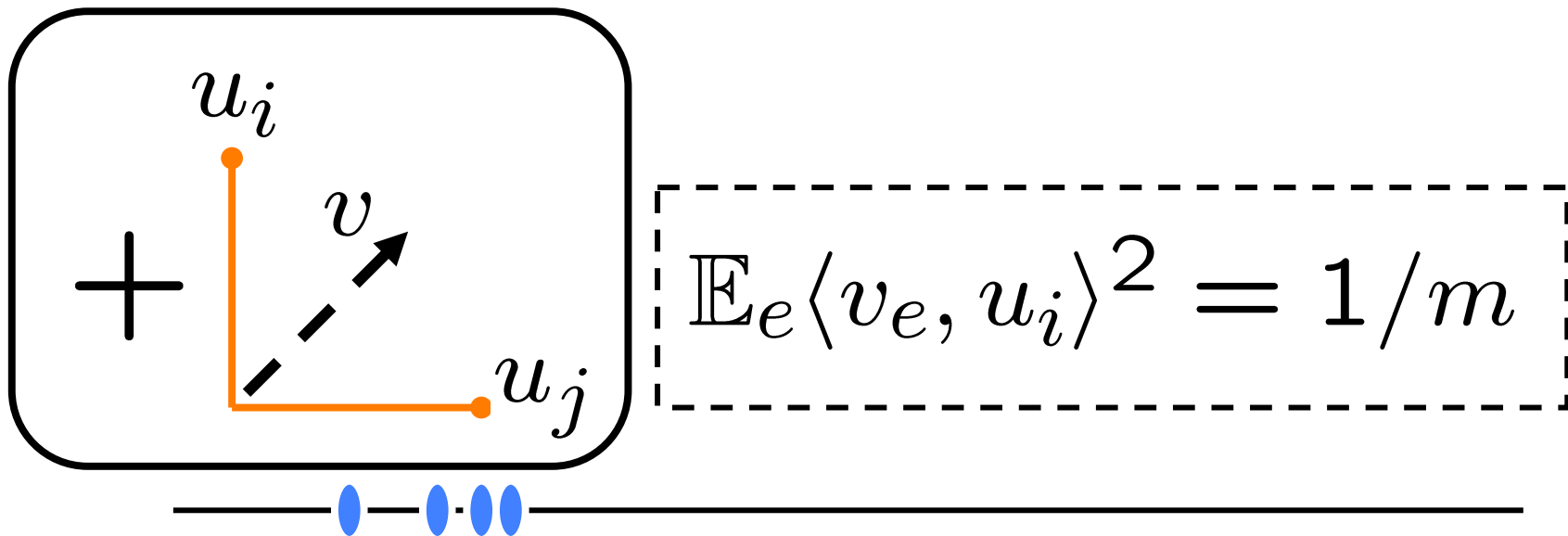
Ideal proof



$$A^{(i+1)} = A^{(i)} + vv^T$$

$$p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n$$

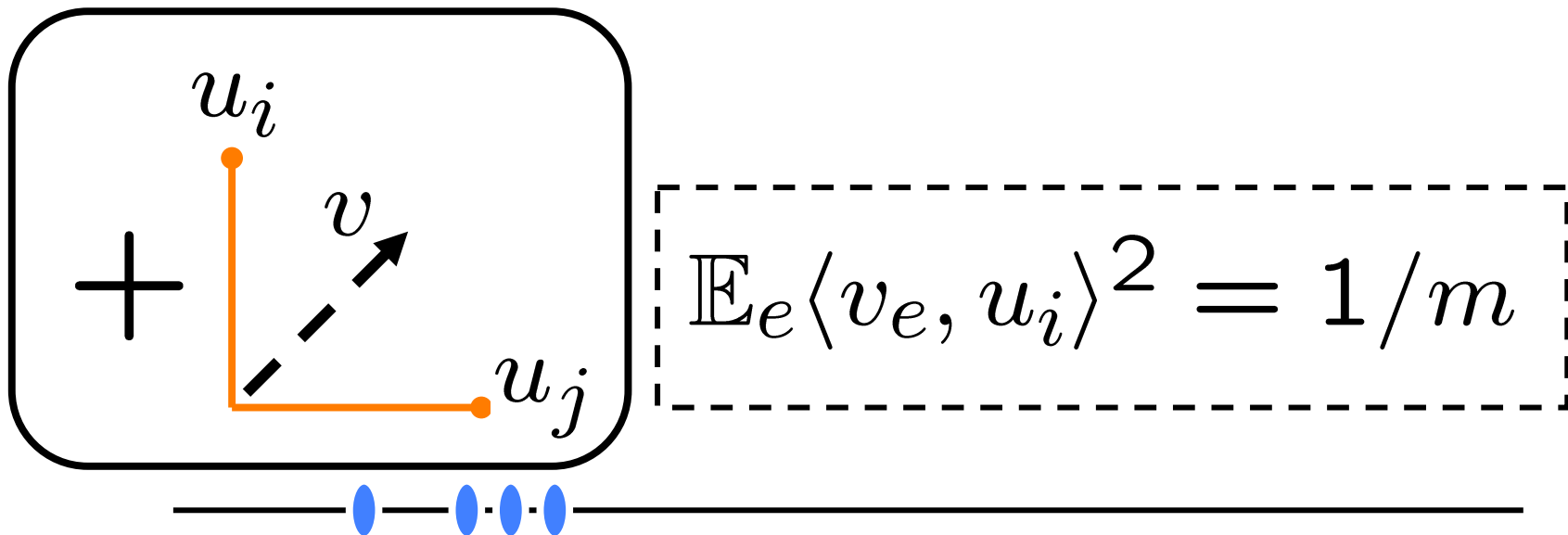
Ideal proof



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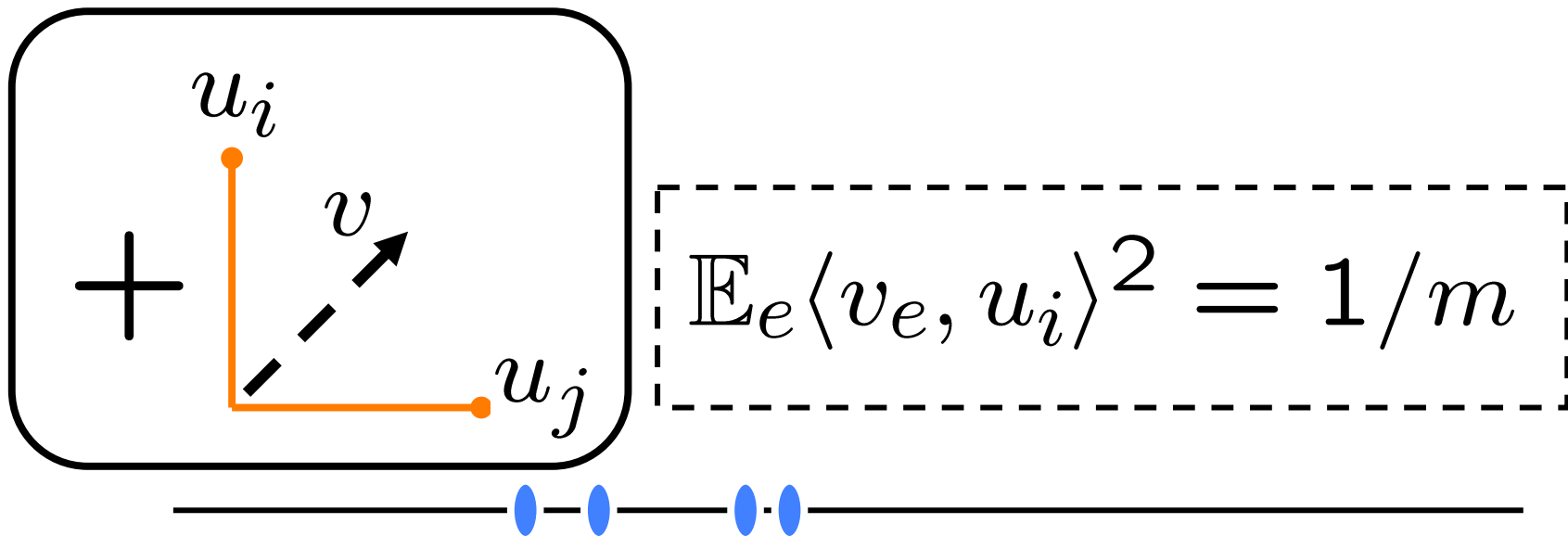
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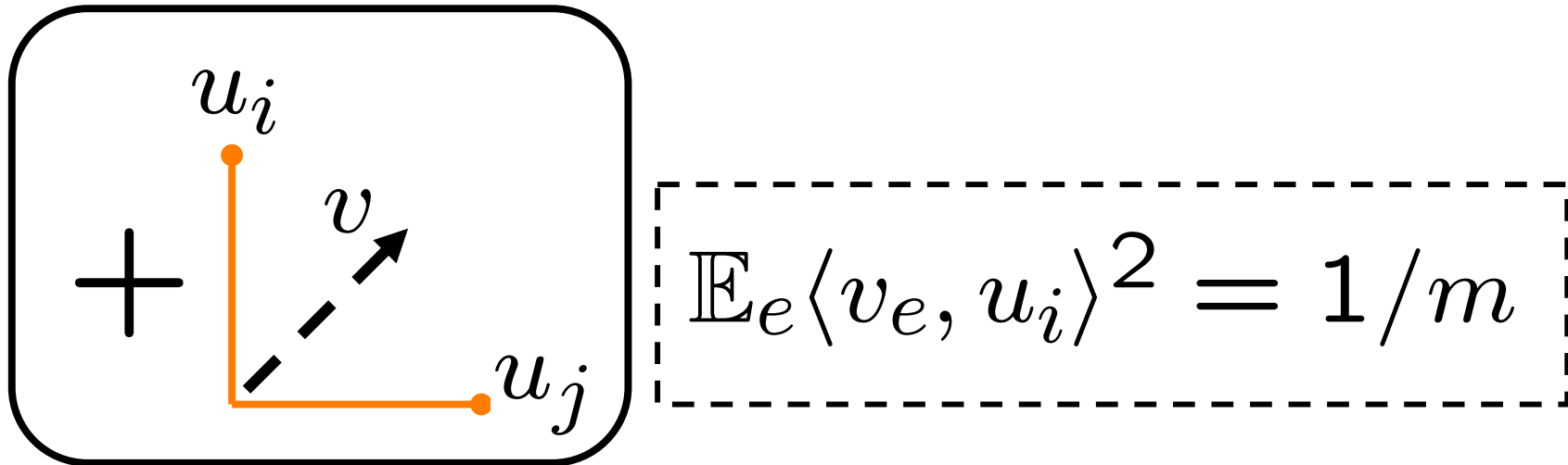
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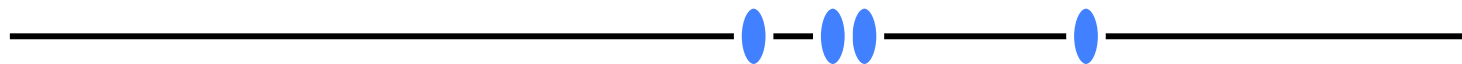
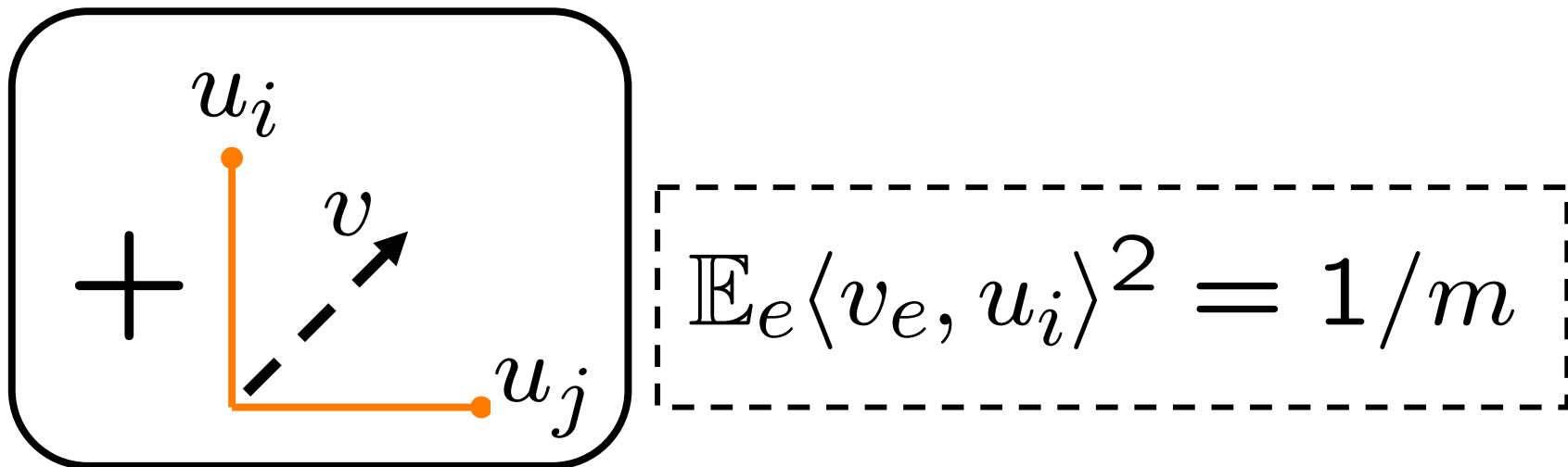
Ideal proof



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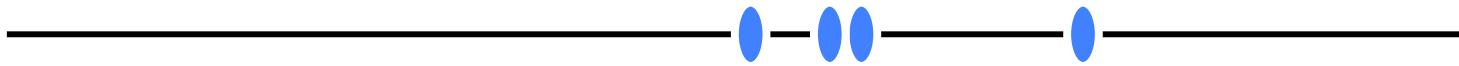
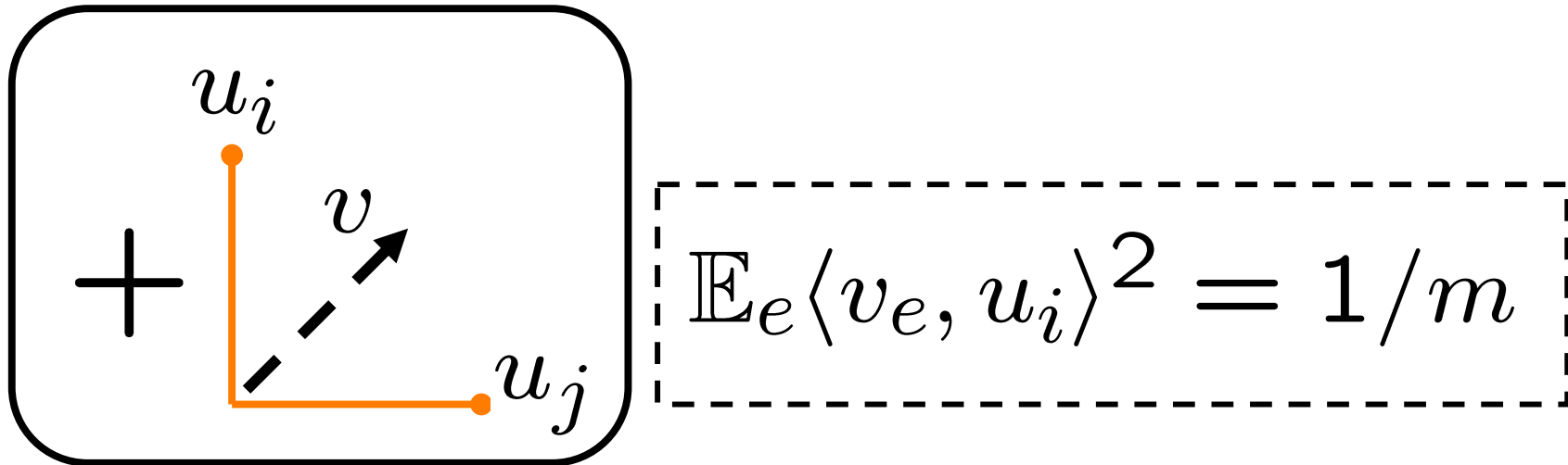
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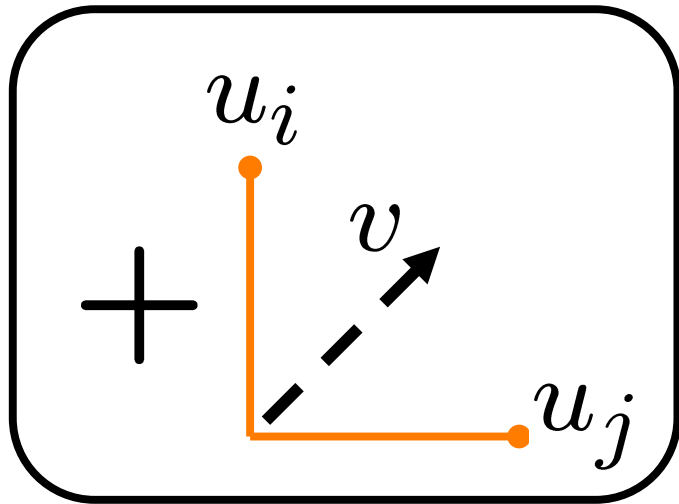
Ideal proof



$$A^{(i+1)} = A^{(i)} + vv^T \quad \dots\dots$$

$$p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n$$

Ideal proof



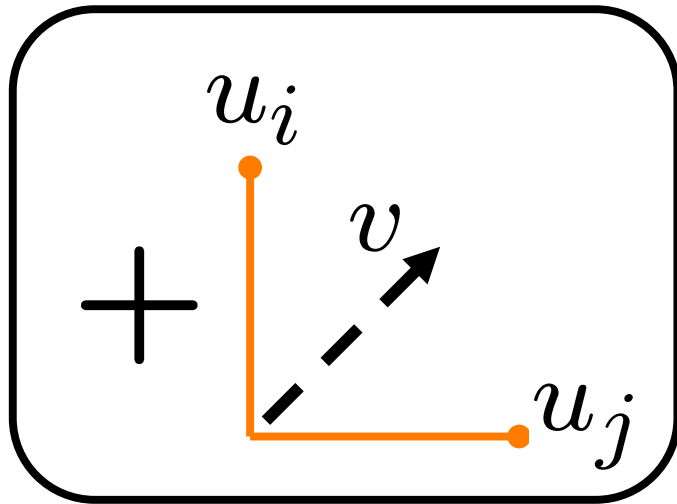
$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



$$A^{(i+1)} = A^{(i)} + vv^T \quad \dots$$

$$p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n$$

Ideal proof



$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



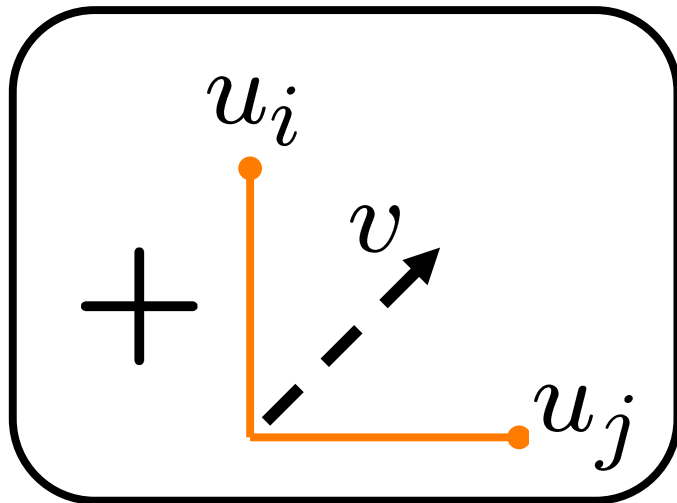
$$A^{(i+1)} = A^{(i)} + vv^T$$

.....

$$p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n$$

$$\frac{\lambda_n(A)}{\lambda_1(A)} \leq 13?$$

Ideal proof



$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



$$A^{(i+1)} = A^{(i)} + vv^T$$

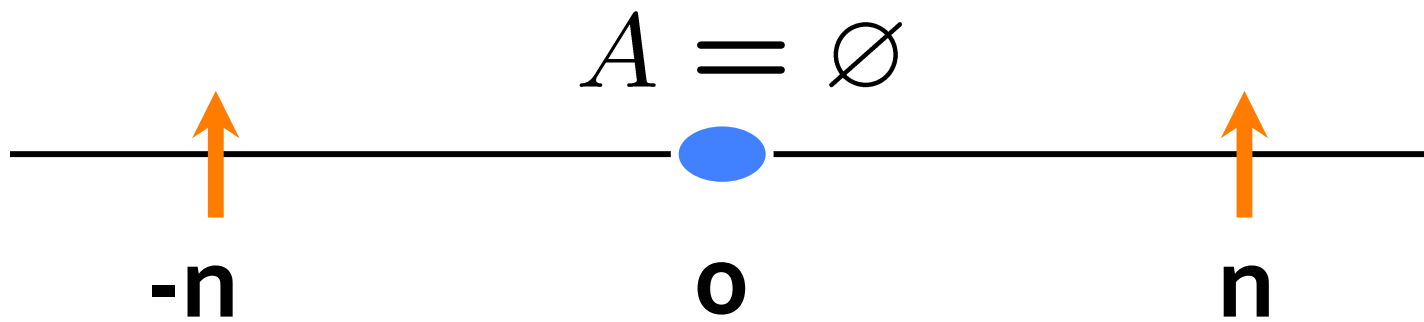
.....

$$p^{(i+1)} = \left(1 - \frac{1}{m} \frac{d}{dx}\right)^{i+1} x^n$$

$$\frac{\lambda_n(A)}{\lambda_1(A)} \rightarrow 1$$

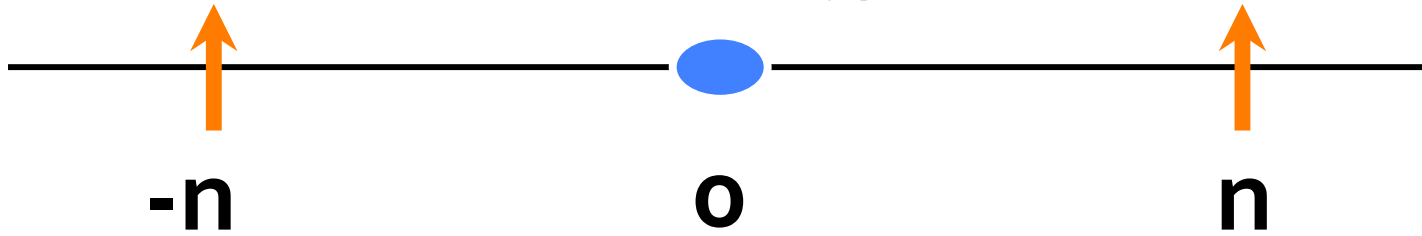
= associated Laguerre polynomial

Broad outline: moving barriers



Step 1

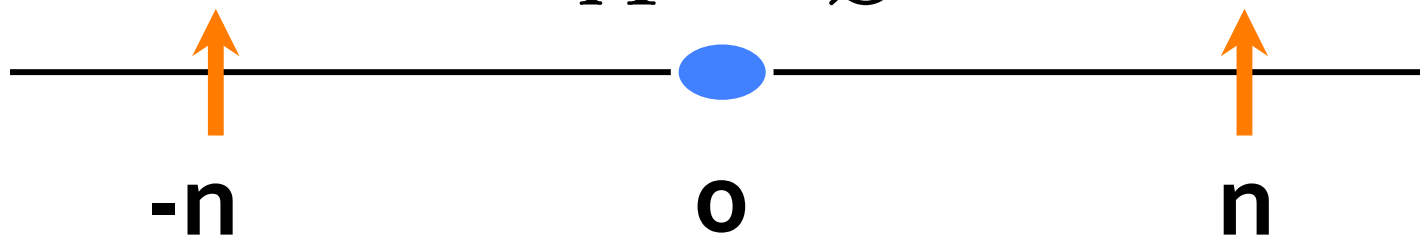
$$A = \emptyset$$



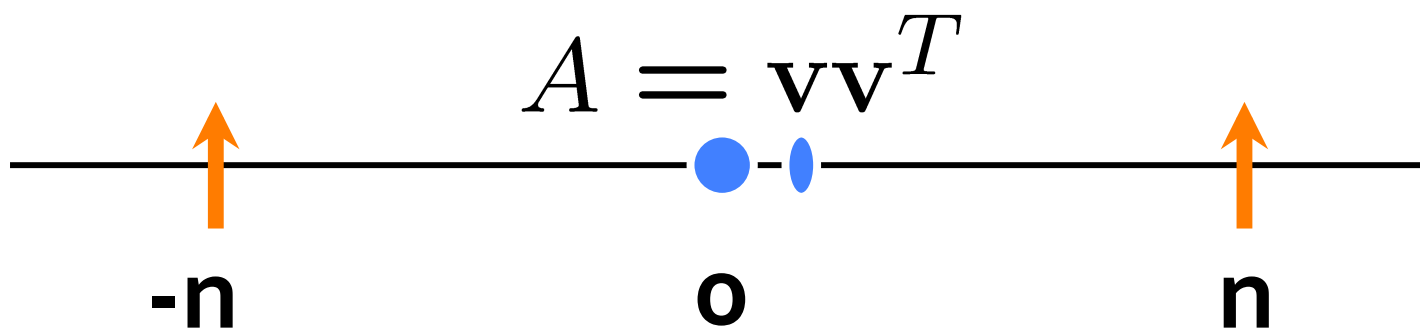
$$+vv^T \quad v \in \{v_e\}$$

Step 1

$$A = \emptyset$$

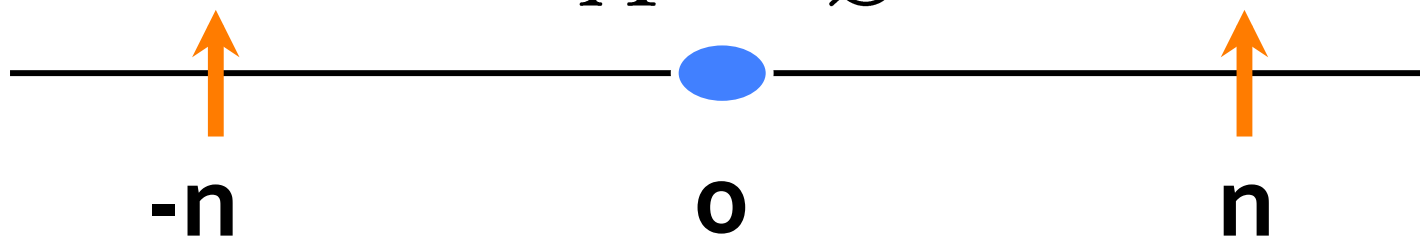


$$+ \mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

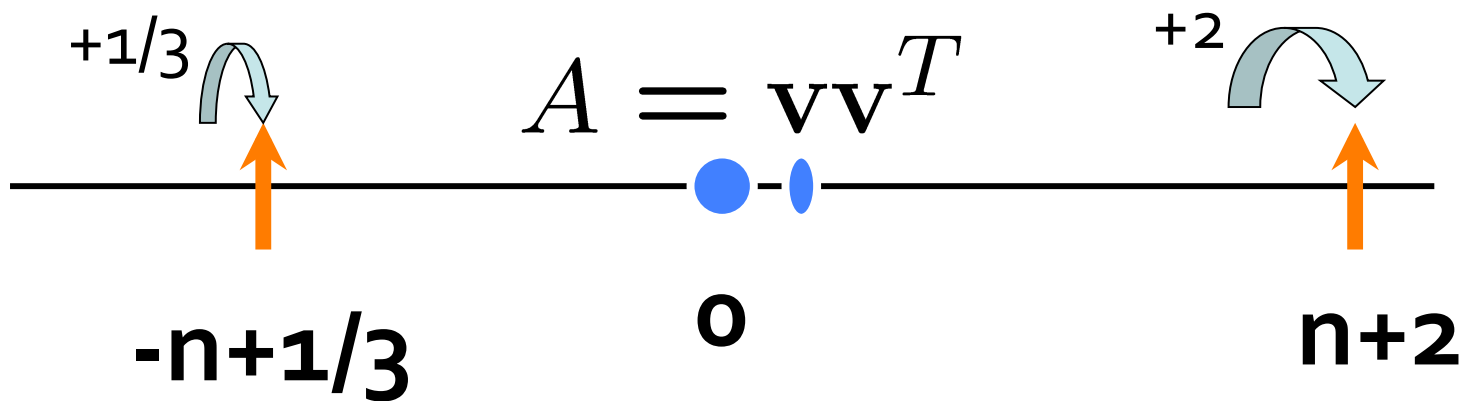


Step 1

$$A = \emptyset$$

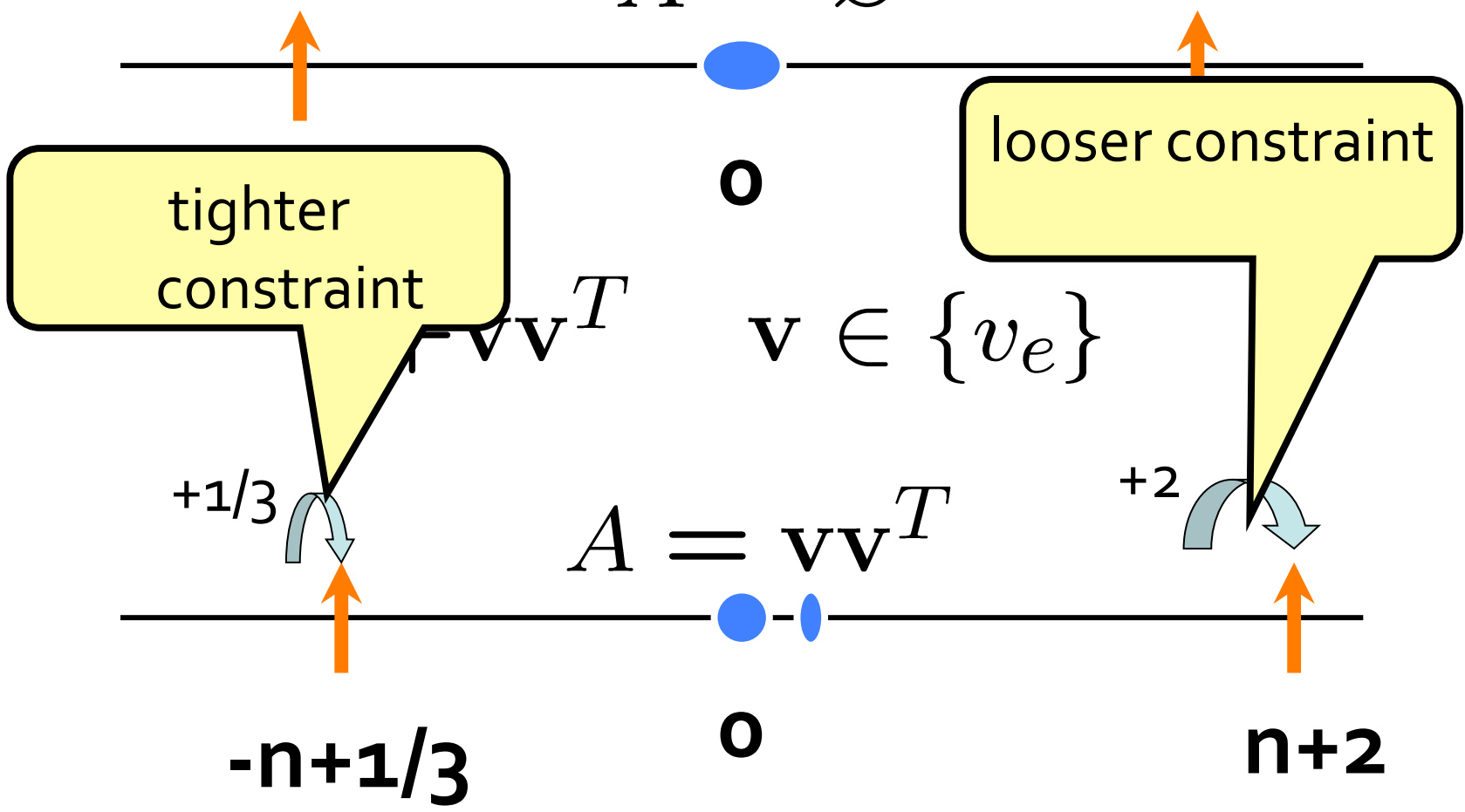


$$+vv^T \quad v \in \{v_e\}$$



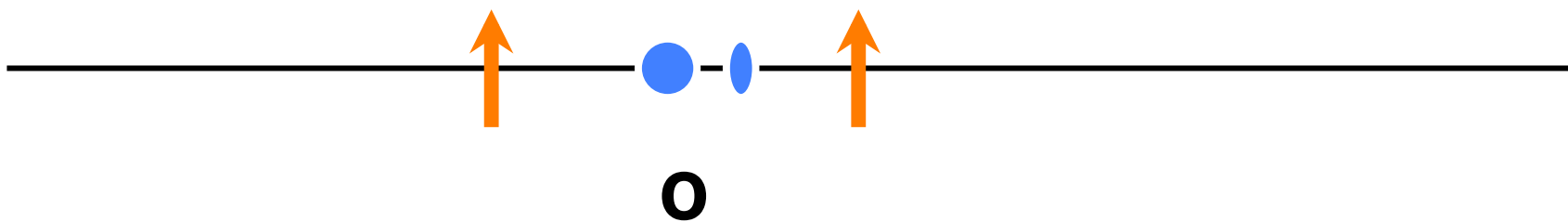
Step 1

$$A = \emptyset$$



Step $i+1$

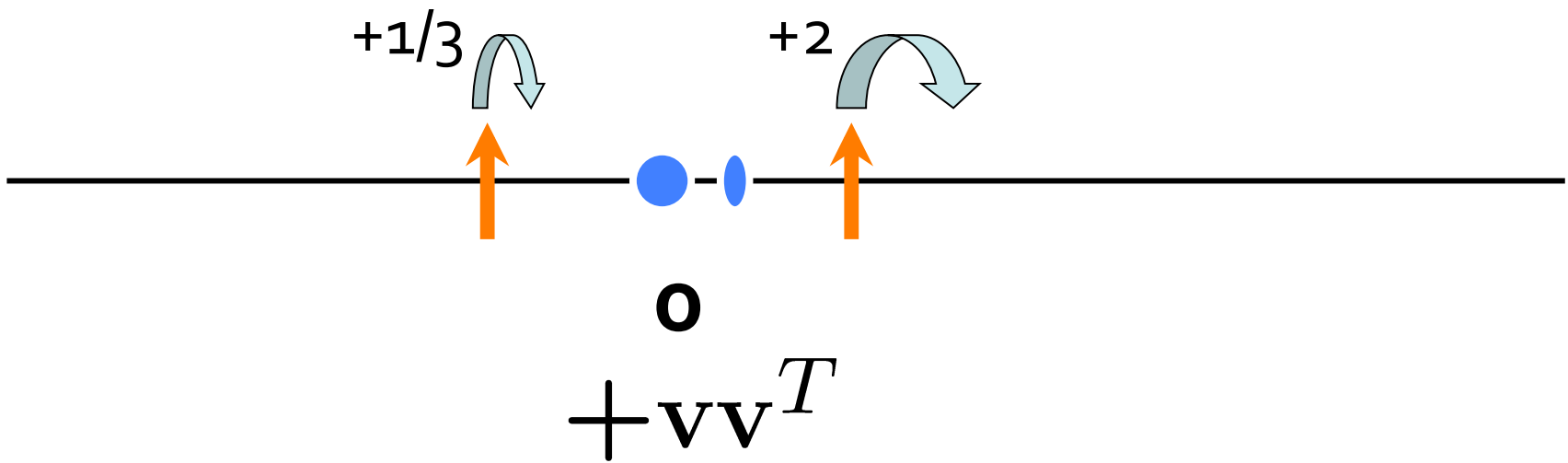
$A^{(i)}$



$$\uparrow \leq \lambda_i \leq \uparrow$$

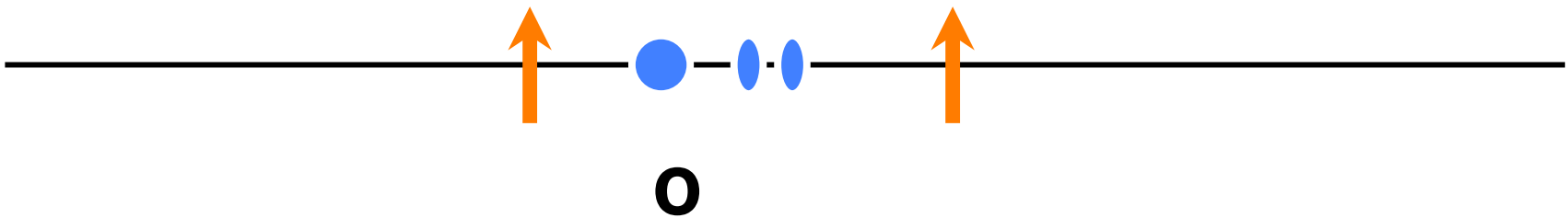
Step $i+1$

$A^{(i)}$



Step $i+1$

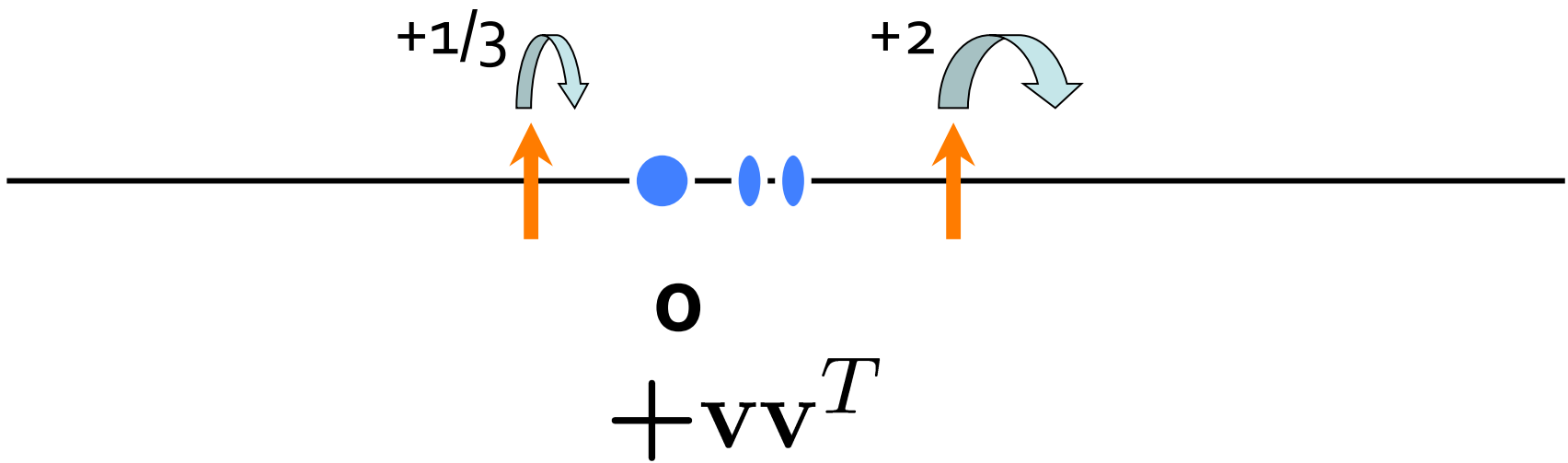
$A^{(i)}, A^{(i+1)}$



$$\uparrow \leq \lambda_i \leq \uparrow$$

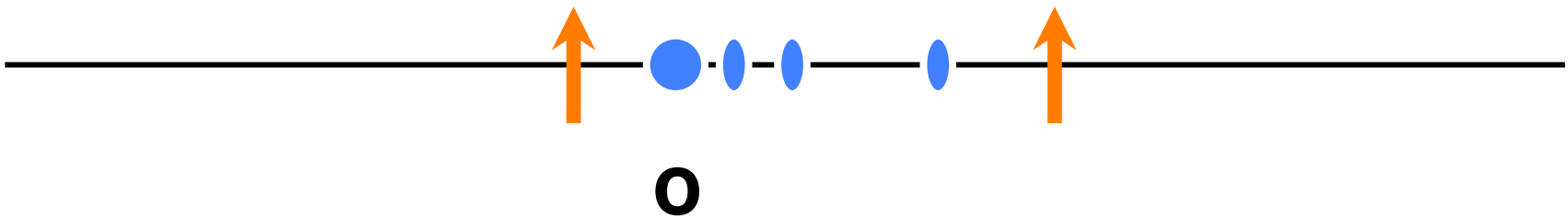
Step $i+1$

$A^{(i)}, A^{(i+1)}$



Step $i+1$

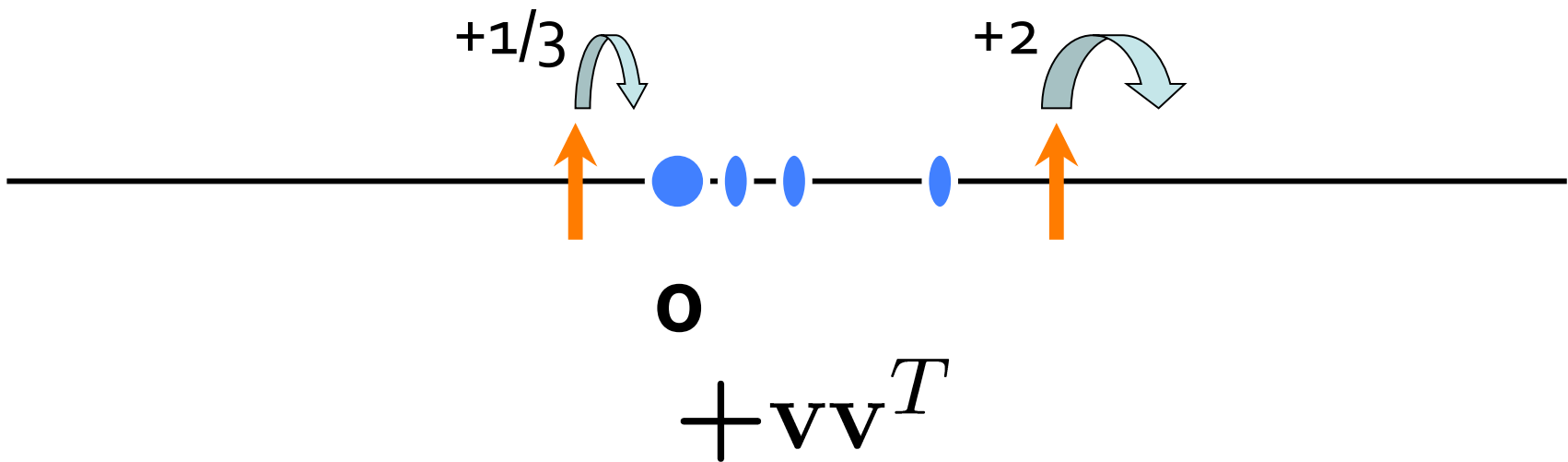
$A^{(i)}, A^{(i+1)}, A^{(i+2)}$



$$\uparrow \leq \lambda_i \leq \uparrow$$

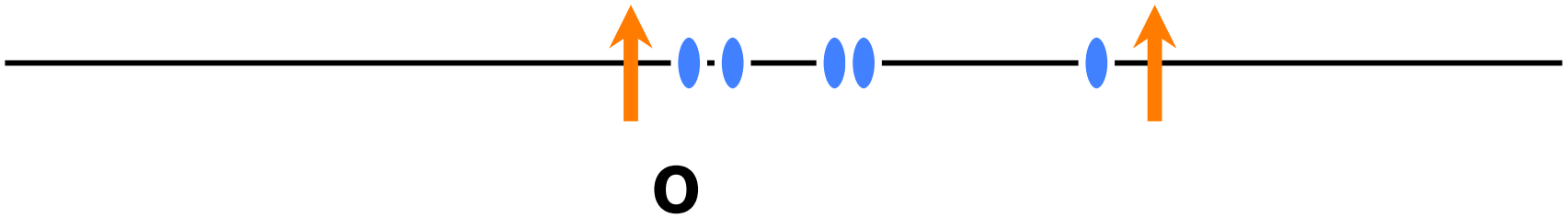
Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}$



Step $i+1$

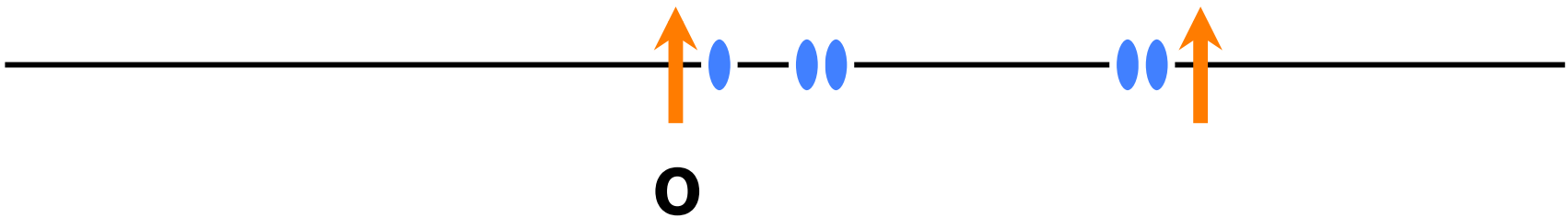
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step $i+1$

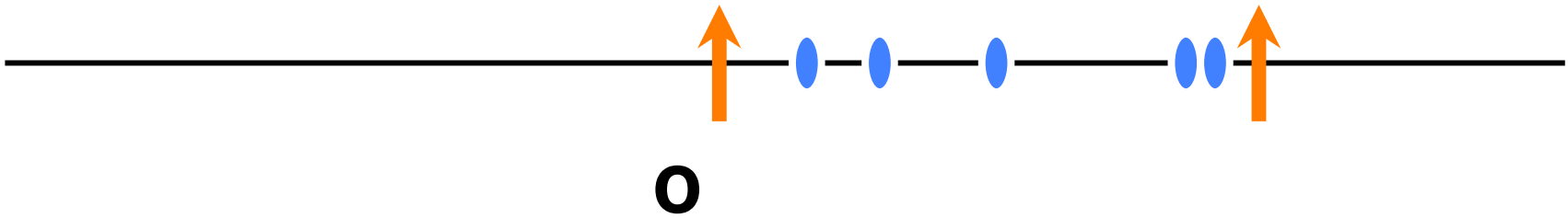
$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step $i+1$

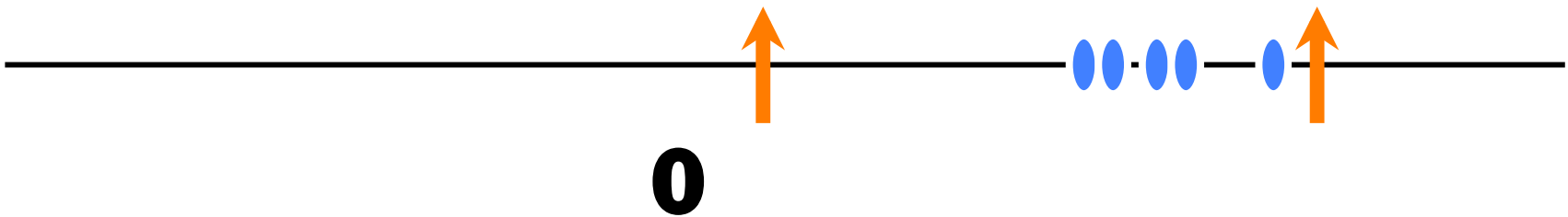
$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step $i+1$

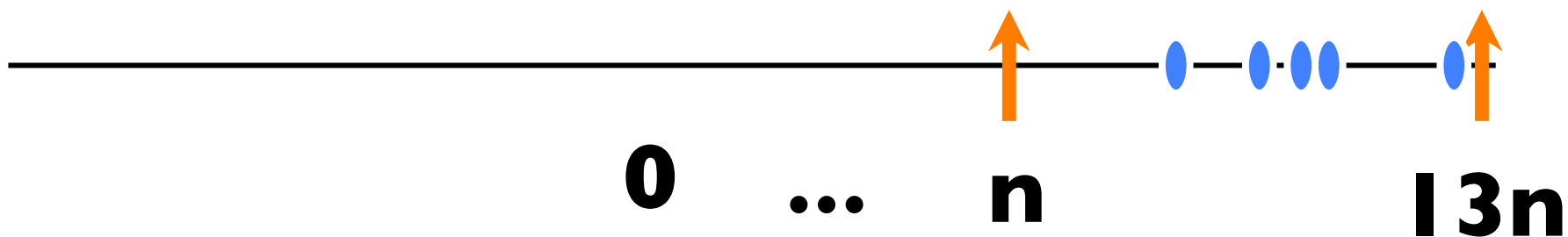
$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step 6n

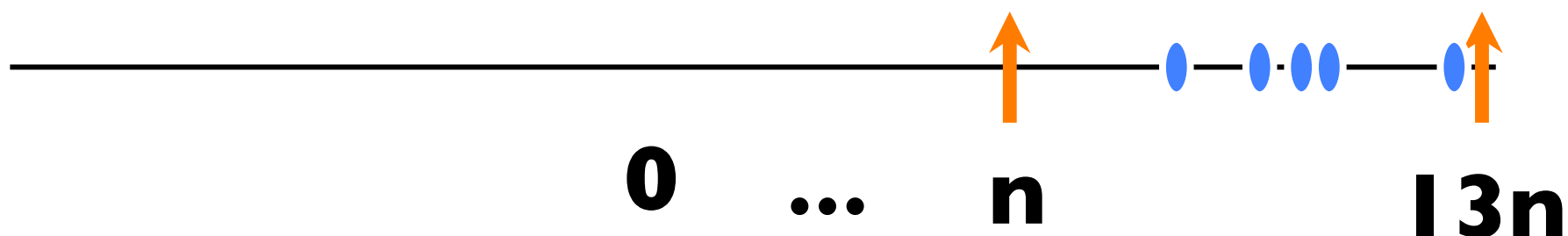
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots, A^{(6n)}$$



$$\uparrow \leq \lambda_i \leq \uparrow$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots, A^{(6n)}$$



2.6-approximation with $6n$ vectors.



Problem

need to show that an appropriate

$$v_e v_e^T$$

always exists.

Problem

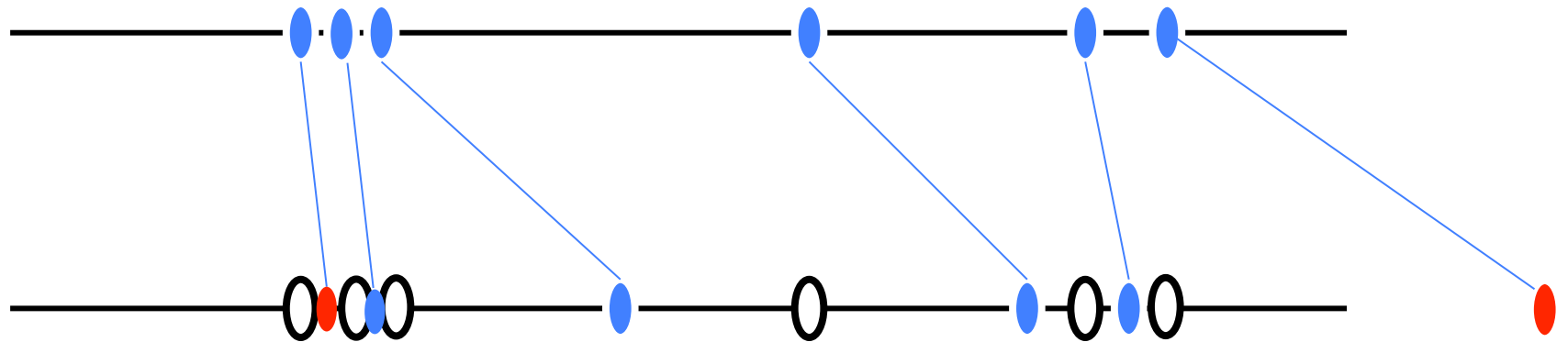
need to show that an appropriate
 $v_e v_e^T$
always exists.

$$\uparrow \leq \lambda_i \leq \uparrow$$

Is not strong enough for induction

Problems

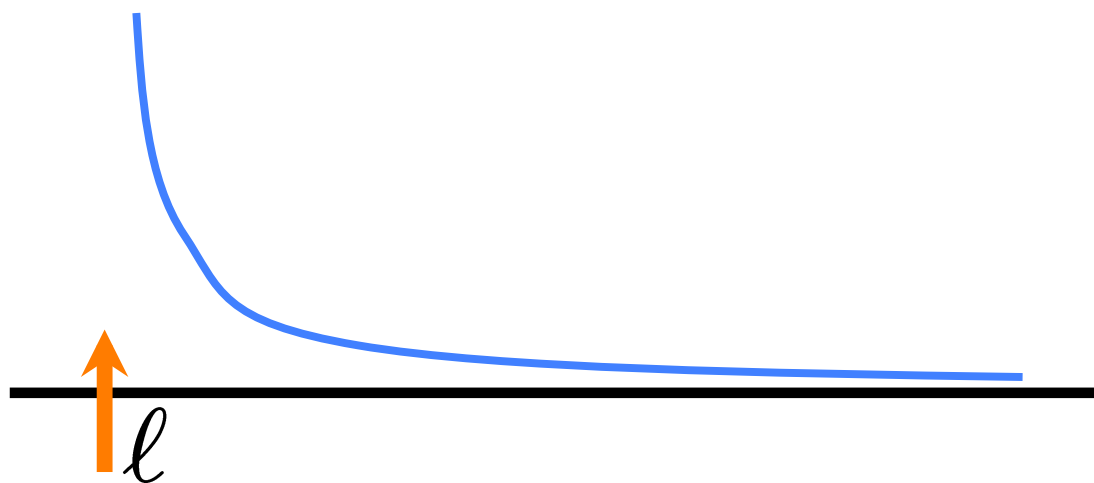
If many small eigs, can only move one



If v_e has large inner product with large eigenvectors, the largest eigenvalue moves far.

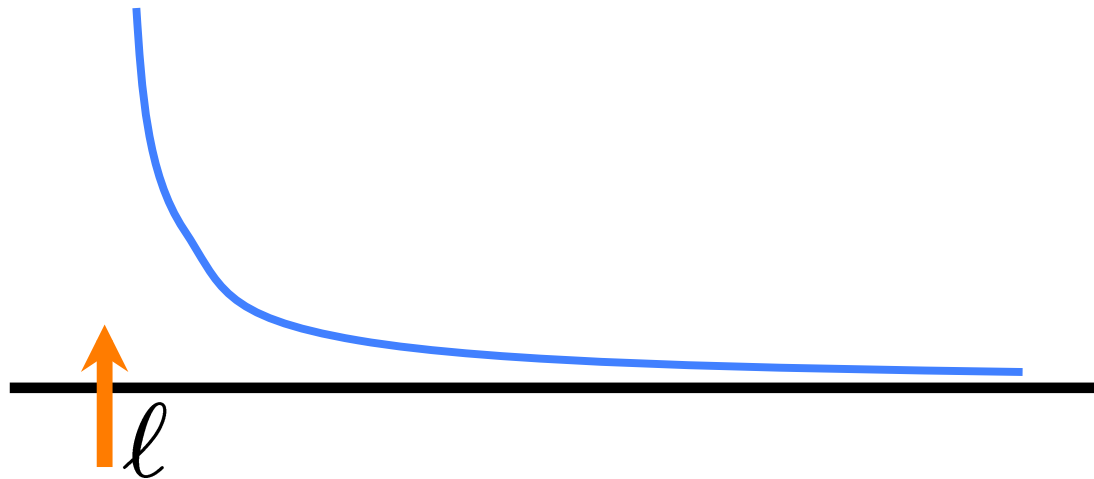
The Lower Barrier Potential Function

$$\Phi_\ell(A) = \sum_i \frac{1}{\lambda_i - \ell} = \text{Tr} \left((A - \ell I)^{-1} \right)$$



The Lower Barrier Potential Function

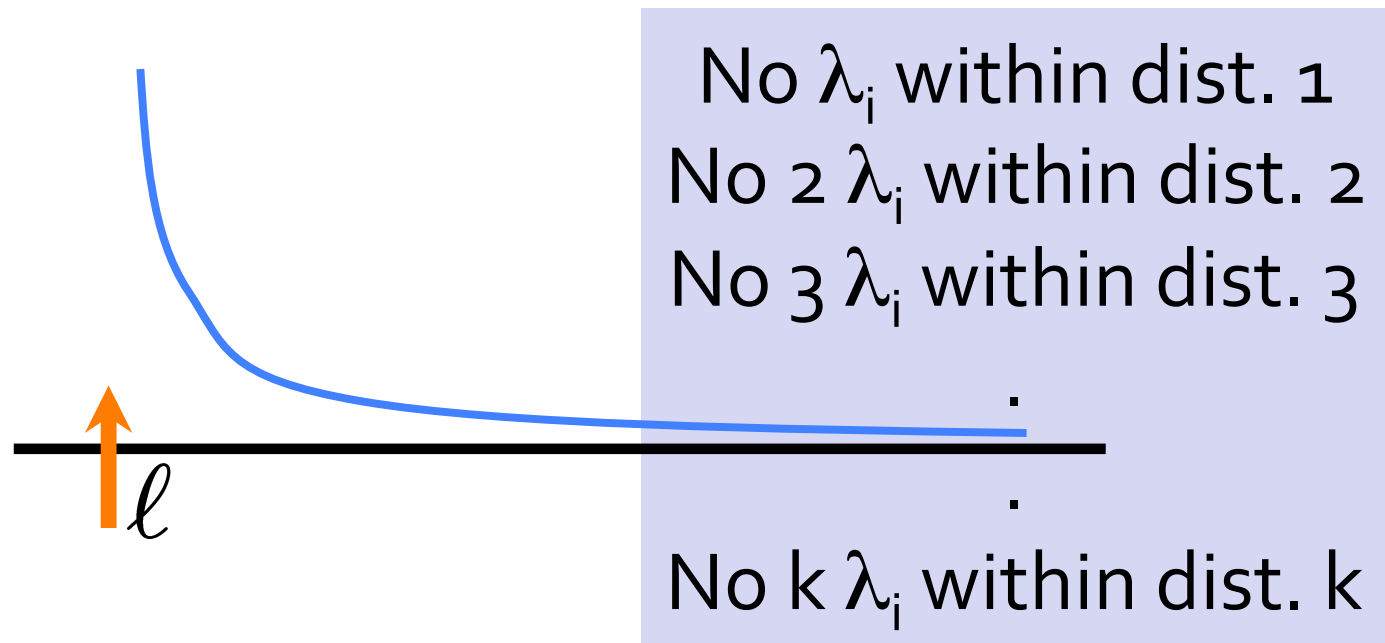
$$\Phi_\ell(A) = \sum_i \frac{1}{\lambda_i - \ell} = \text{Tr} \left((A - \ell I)^{-1} \right)$$



$$\Phi_\ell(A) \leq 1 \implies \lambda_{\min}(A) \geq \ell + 1$$

The Lower Barrier Potential Function

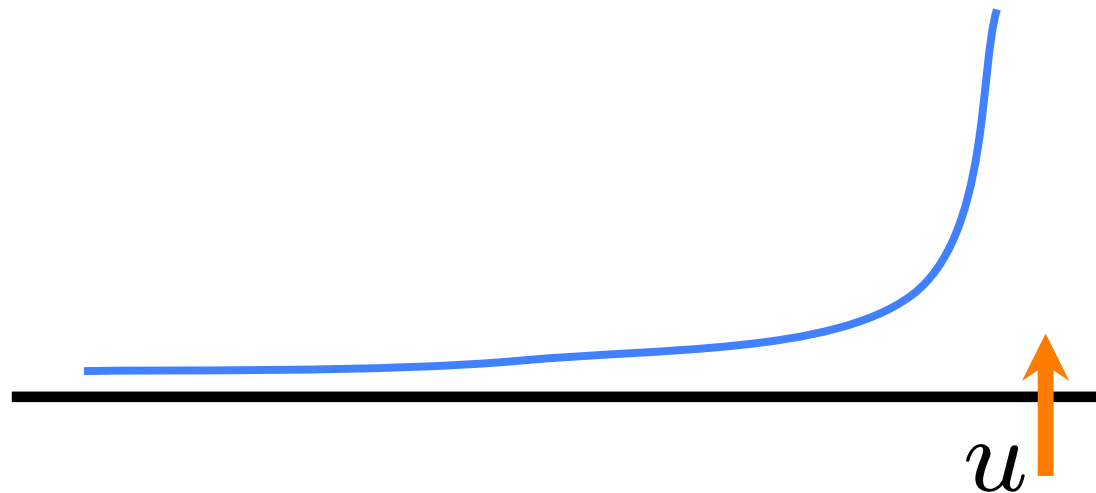
$$\Phi_\ell(A) = \sum_i \frac{1}{\lambda_i - \ell} = \text{Tr} \left((A - \ell I)^{-1} \right)$$



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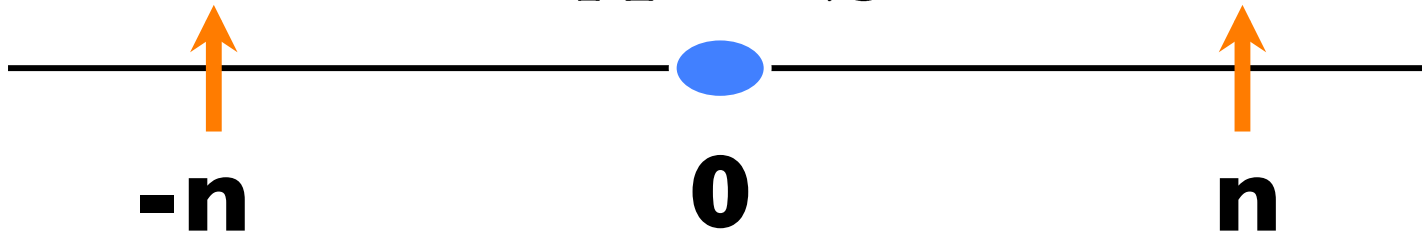
The Upper Barrier Potential Function

$$\Phi^u(A) = \sum_i \frac{1}{u - \lambda_i} = \text{Tr} \left((uI - A)^{-1} \right)$$



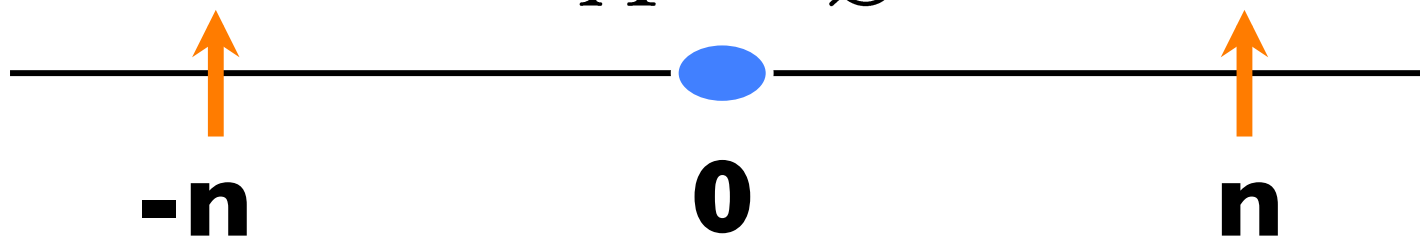
The Beginning

$$A = \emptyset$$



The Beginning

$$A = \emptyset$$

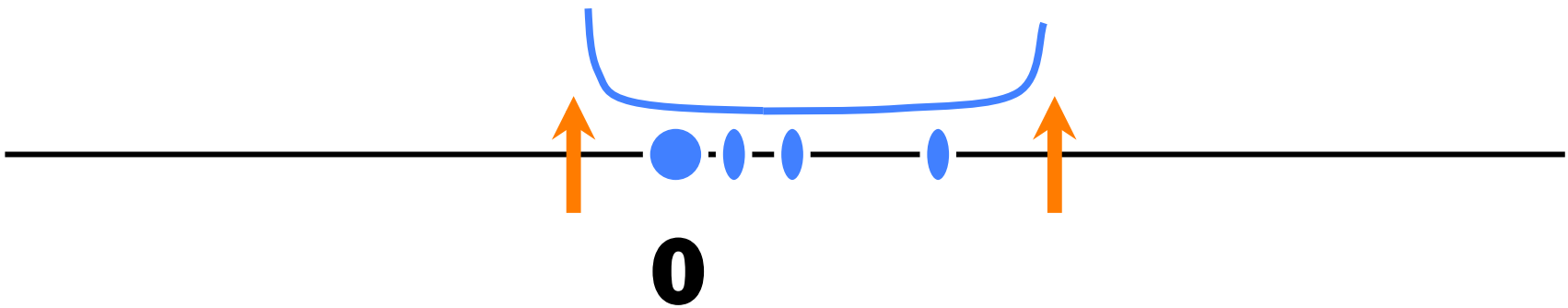


$$\Phi^n(\emptyset) = \text{Tr}(nI)^{-1} = 1$$

$$\Phi_{-n}(\emptyset) = \text{Tr}(nI)^{-1} = 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}$

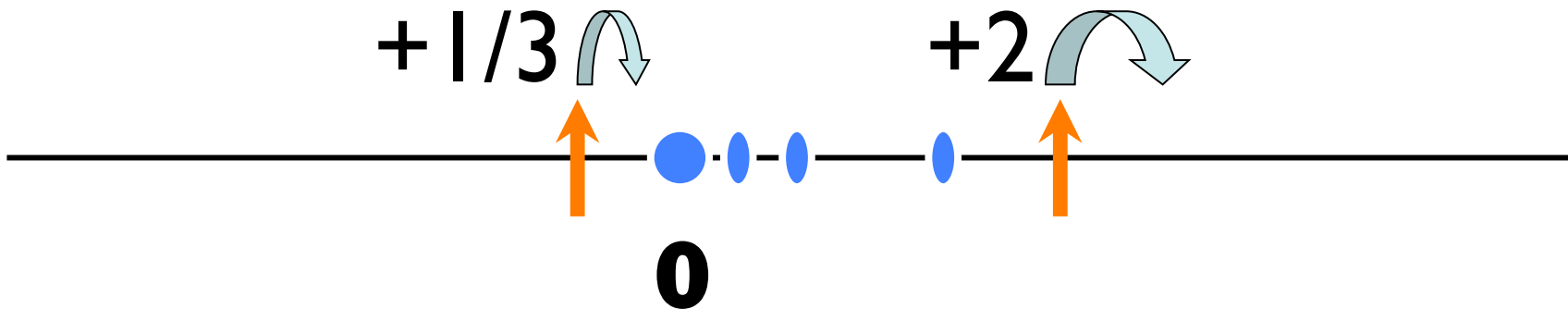


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}$

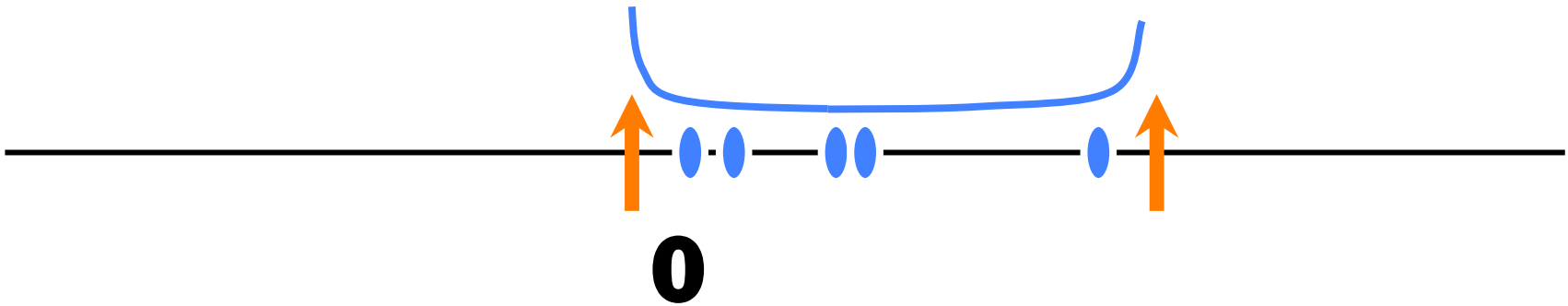


Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$ $\Phi^u(A) \leq 1$
so that potentials do not increase $\Phi_\ell(A) \leq 1.$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$

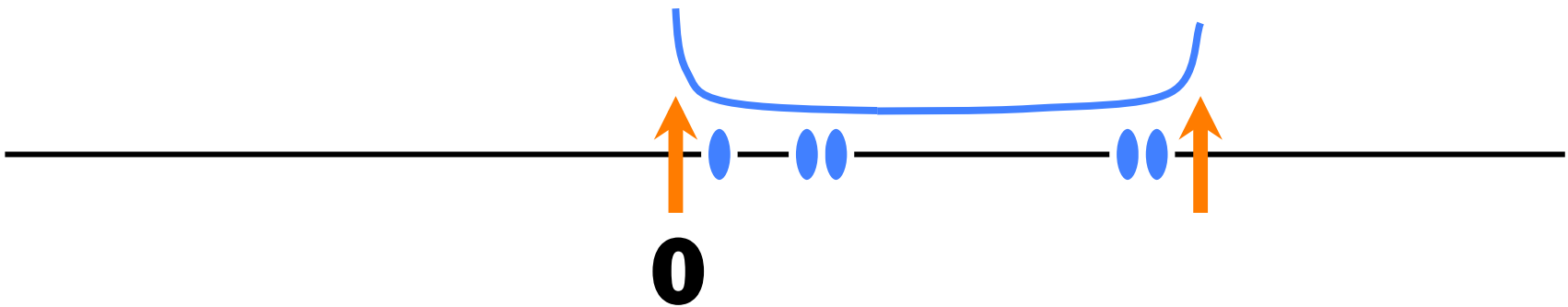


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$

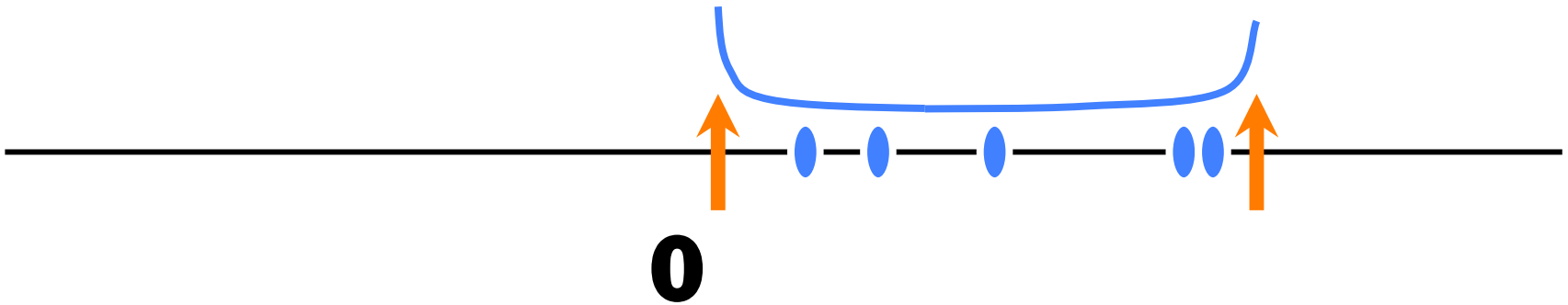


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$

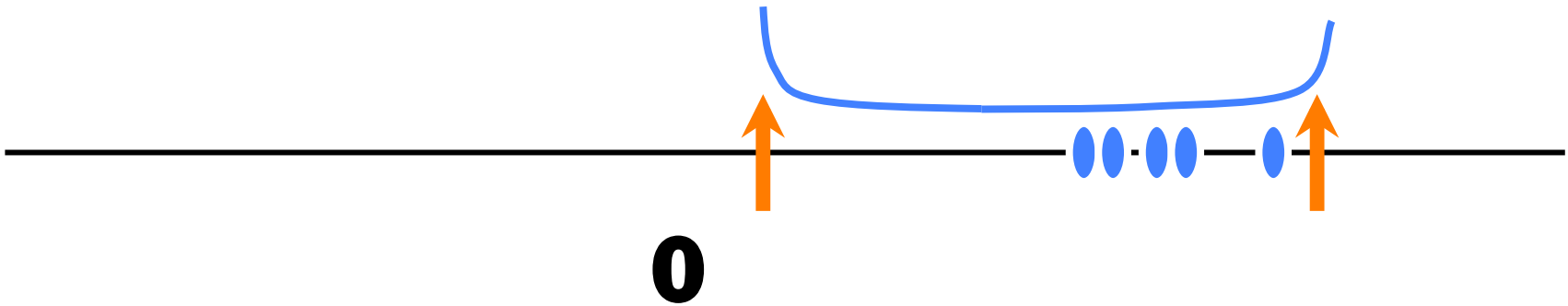


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$



$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots, A^{(6n)}$

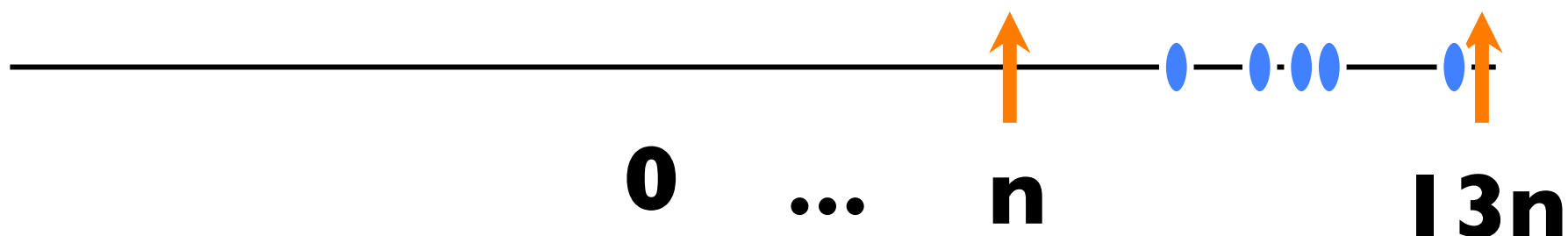


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots, A^{(6n)}$$



2.6-approximation with $6n$ vectors.



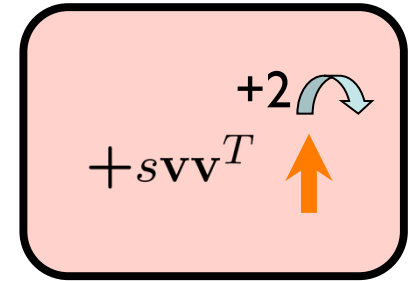
Goal

Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$ $\Phi^u(A) \leq 1$
so that potentials do not increase $\Phi_\ell(A) \leq 1$.

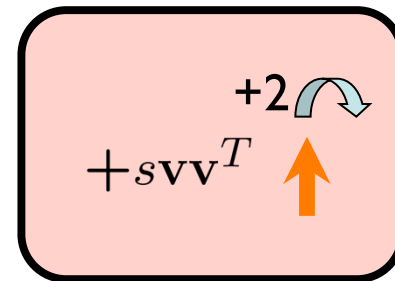
$$\begin{array}{ccc} +1/3 & \curvearrowright & +2 \\ \uparrow & & \uparrow \\ +s\mathbf{v}\mathbf{v}^T & & \end{array}$$

Upper Barrier Update



Add svv^T and set $u' \leftarrow u + 2$.

Upper Barrier Update

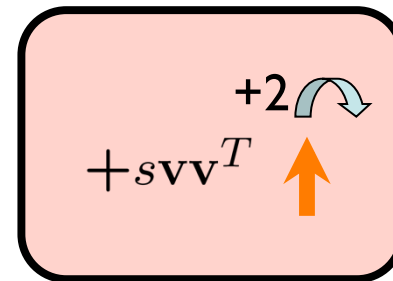


Add svv^T and set $u' \leftarrow u + 2$.

$$\Phi^{u'}(A + svv^T)$$

$$= \text{Tr} \left((u'I - A - svv^T)^{-1} \right)$$

Upper Barrier Update

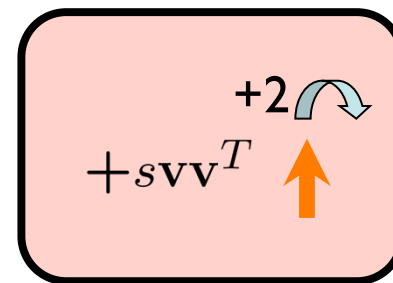


Add svv^T and set $u' \leftarrow u + 2$.

$$\begin{aligned} & \Phi^{u'}(A + svv^T) \\ &= \text{Tr} \left((u'I - A - svv^T)^{-1} \right) \\ &= \Phi^{u'}(A) + \frac{sv^T (u'I - A)^{-2} v}{1 - sv^T (u'I - A)^{-1} v} \end{aligned}$$

By Sherman-Morrison Formula

Upper Barrier Update



Add svv^T and set $u' \leftarrow u + 2$.

$$\begin{aligned} & \Phi^{u'}(A + svv^T) \\ &= \text{Tr} \left((u'I - A - svv^T)^{-1} \right) \\ &= \Phi^{u'}(A) + \frac{sv^T (u'I - A)^{-2} v}{1 - sv^T (u'I - A)^{-1} v} \end{aligned}$$

$$\text{Need } \leq \Phi^u(A)$$

How much of $\mathbf{v}\mathbf{v}^T$ can we add?

Rearranging:

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

iff

$$1 \geq s\mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^u(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

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Rearranging:

$$\Phi^{u'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi^u(A)$$

iff

$$1 \geq s\mathbf{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^u(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \mathbf{v}$$

Write as

$$1 \geq s\mathbf{v}^T U_A \mathbf{v}$$

Lower Barrier

Similarly:

$$\Phi_{l'}(A + s\mathbf{v}\mathbf{v}^T) \leq \Phi_l(A)$$

iff

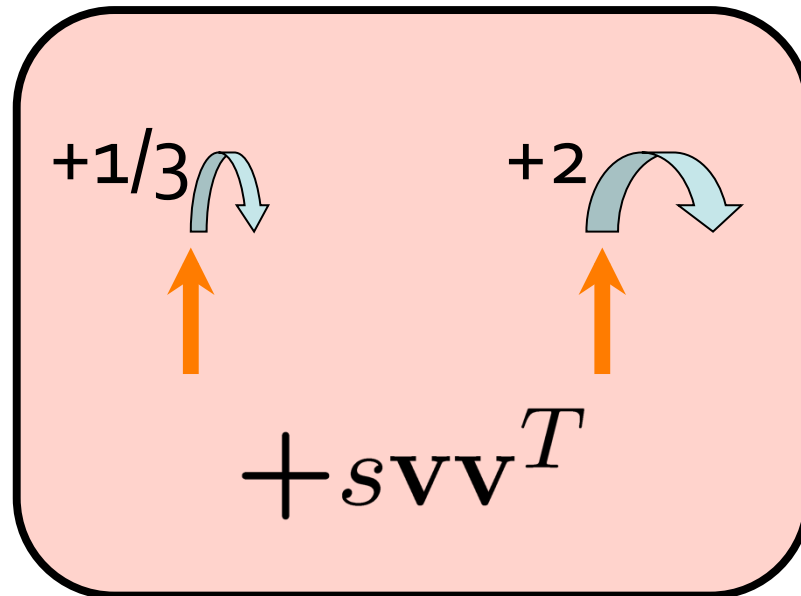
$$1 \leq s\mathbf{v}^T \left(\frac{(A - l'I)^{-2}}{\Phi_{l'}(A) - \Phi_l(A)} - (A - l'I)^{-1} \right) \mathbf{v}$$

Write as

$$1 \leq s\mathbf{v}^T L_A \mathbf{v}$$

Goal

Show that we can always add some vector while respecting *both* barriers.



Need: $sv^T U_A v \leq 1 \leq sv^T L_A v$

Two expectations

Need: $s\mathbf{v}^T U_A \mathbf{v} \leq 1 \leq s\mathbf{v}^T L_A \mathbf{v}$

Can show: $\mathbf{E}_e \left[\mathbf{v}_e^T U_A \mathbf{v}_e \right] \leq 1$

$$\mathbf{E}_e \left[\mathbf{v}_e^T L_A \mathbf{v}_e \right] \geq 1$$

Two expectations

Need: $s\mathbf{v}^T U_A \mathbf{v} \leq 1 \leq s\mathbf{v}^T L_A \mathbf{v}$

Can show: $\mathbf{E}_e \left[\mathbf{v}_e^T U_A \mathbf{v}_e \right] \leq 1$

$$\mathbf{E}_e \left[\mathbf{v}_e^T L_A \mathbf{v}_e \right] \geq 1$$

So: $\mathbf{E}_e \left[\mathbf{v}_e^T U_A \mathbf{v}_e \right] \leq \mathbf{E}_e \left[\mathbf{v}_e^T L_A \mathbf{v}_e \right]$

And, exists $e : \mathbf{v}_e^T U_A \mathbf{v}_e \leq \mathbf{v}_e^T L_A \mathbf{v}_e$

Two expectations

Need: $s \mathbf{v}^T U_A \mathbf{v} \leq 1 \leq s \mathbf{v}^T L_A \mathbf{v}$

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And, exists $e : \mathbf{v}_e^T U_A \mathbf{v}_e \leq \mathbf{v}_e^T L_A \mathbf{v}_e$

And s that puts 1 between them

Bounding expectations

$$\mathbf{v}^T U_A \mathbf{v} = U_A \bullet \mathbf{v} \mathbf{v}^T$$

$$\begin{aligned} \mathbf{E}_e \left[U_A \bullet \mathbf{v}_e \mathbf{v}_e^T \right] &= U_A \bullet \mathbf{E}_e \left[\mathbf{v}_e \mathbf{v}_e^T \right] \\ &= U_A \bullet I \\ &= \text{Tr} (U_A) \end{aligned}$$

Bounding expectations

$$\mathrm{Tr}(U_A) = \frac{\mathrm{Tr}((u'I - A)^{-2})}{\Phi^u(A) - \Phi^{u'}(A)} + \mathrm{Tr}((u'I - A)^{-1})$$

$$\uparrow \leq \frac{1}{u' - u}$$

$$\uparrow = \Phi^{u'}(A)$$

$$\leq \frac{1}{2} + 1 = \frac{3}{2}$$

Bounding expectations

Similarly,

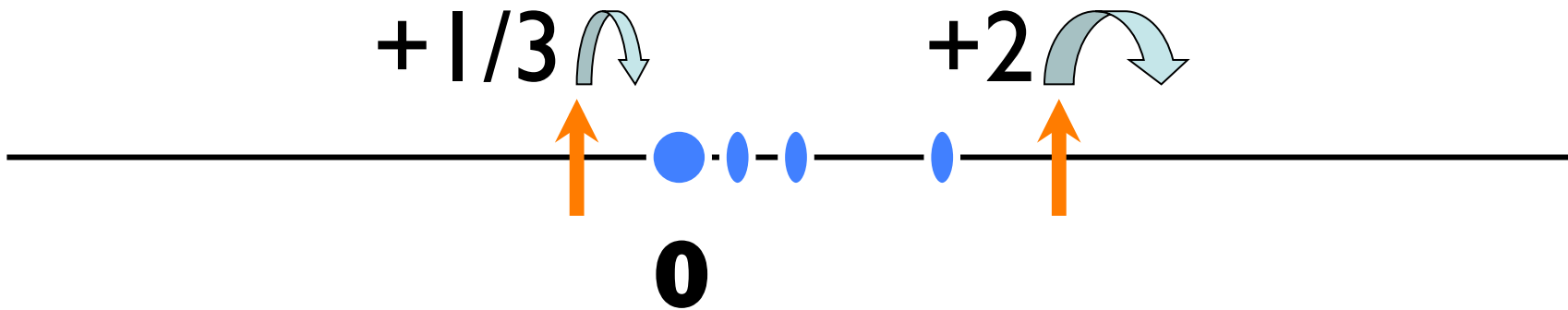
$$\begin{aligned}\mathrm{Tr}(L_A) &\geq \frac{1}{l' - l} - 1 \\ &= \frac{1}{1/3} - 1 \\ &= 2\end{aligned}$$

So

$$\mathbf{E}_e \left[\mathbf{v}_e^T U_A \mathbf{v}_e \right] = \mathrm{Tr}(U_A) \leq \mathrm{Tr}(L_A) = \mathbf{E}_e \left[\mathbf{v}_e^T L_A \mathbf{v}_e \right]$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}$

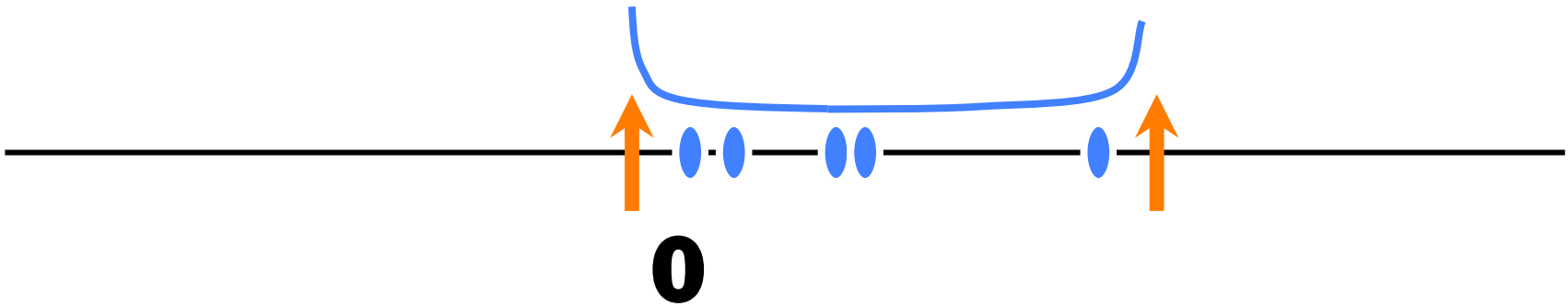


Lemma.

can always choose $+s\mathbf{v}\mathbf{v}^T$ $\Phi^u(A) \leq 1$
so that potentials do not increase $\Phi_\ell(A) \leq 1.$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$

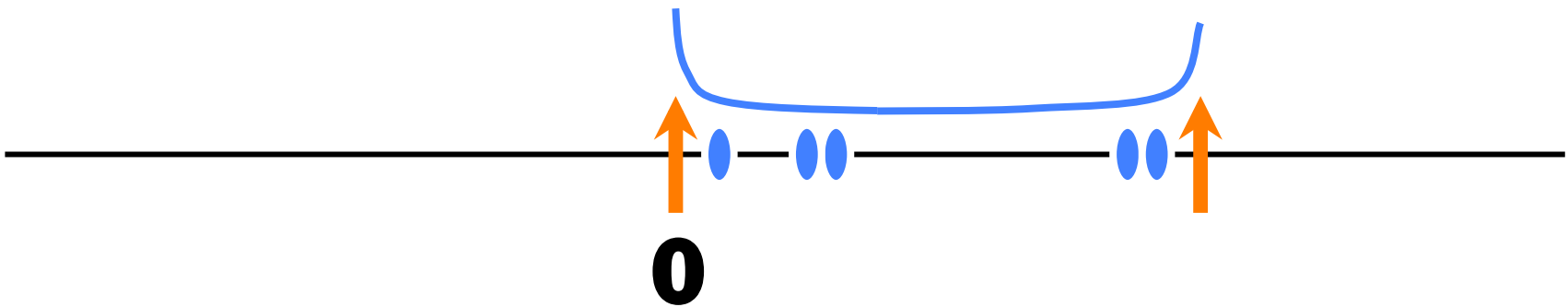


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$

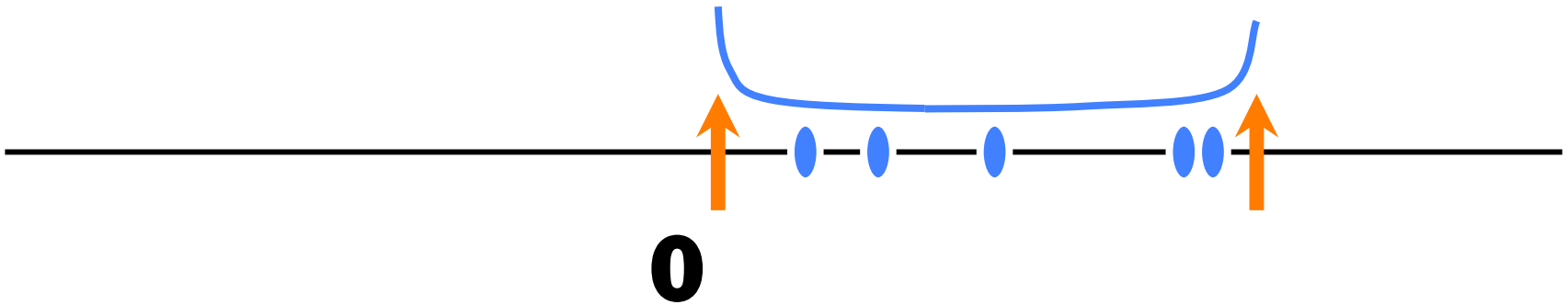


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$

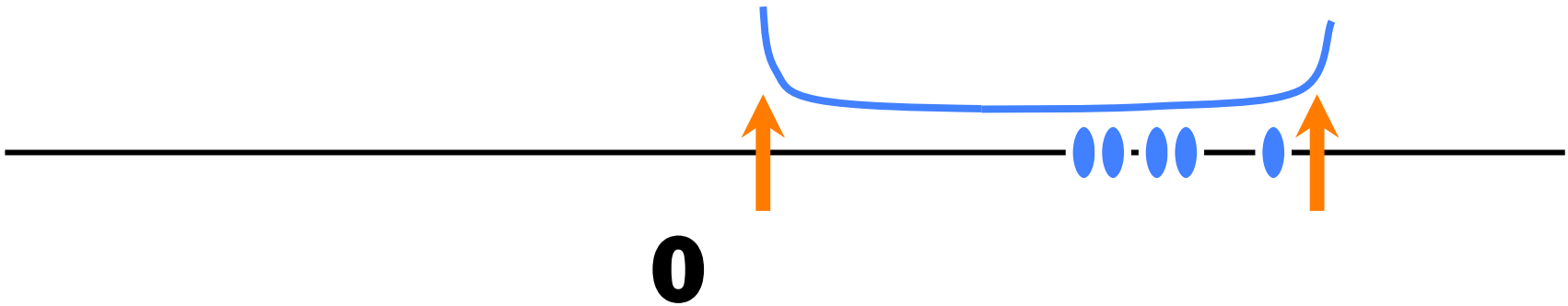


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step $i+1$

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots$



$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots, A^{(6n)}$

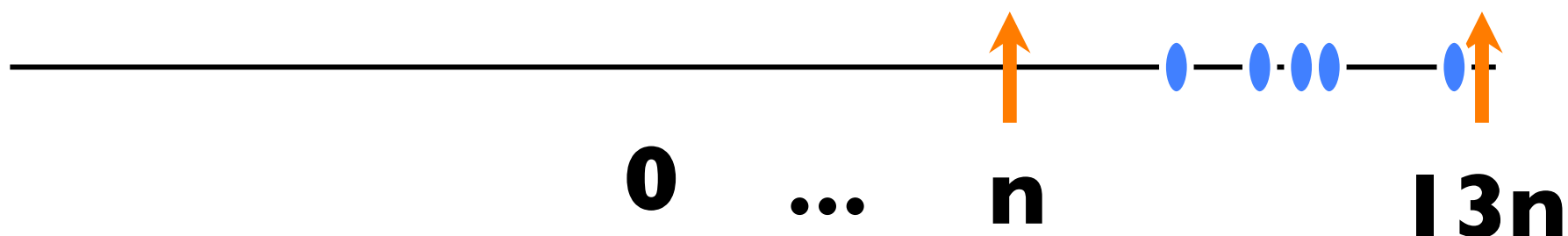


$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \dots, A^{(6n)}$$



2.6-approximation with $6n$ vectors.



Twice-Ramanujan

Fixing dn steps and tightening parameters
gives ratio

$$\frac{\lambda_{max}(A)}{\lambda_{min}(A)} \leq \frac{d + 1 + 2\sqrt{d}}{d + 1 - 2\sqrt{d}}$$

Less than twice as many edges as used by
Ramanujan Expander of same quality

Open Questions

- Unweighted sparsifiers of complete graph.
- The Ramanujan bound
- Properties of vectors from graphs?
- Faster algorithm
 - union of random Hamiltonian cycles?
- The Kadison-Singer Conjecture
 - Implied if $s_e \in \{0, 1\}$ when $\|v_e\|$ constant
 - Achieved by random sampling, but $O(n \log n)$ vecs.