These are the slides from the talk that I gave on sparsification at EPFL on June 11, 2012.

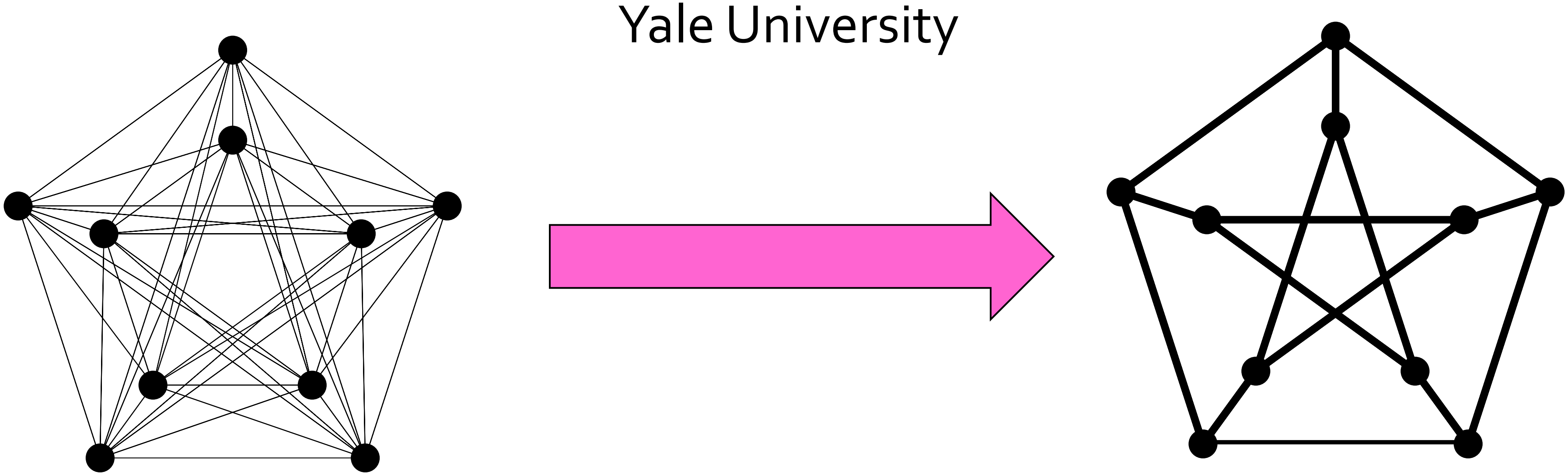
Since I did not get through all the slides during the talk, I've added some comments.

I'm reserving this font for material that did not appear in the slides.

--Dan Spielman

Spectral Sparsification of Graphs and Approximations of Matrices

Daniel A. Spielman
Yale University



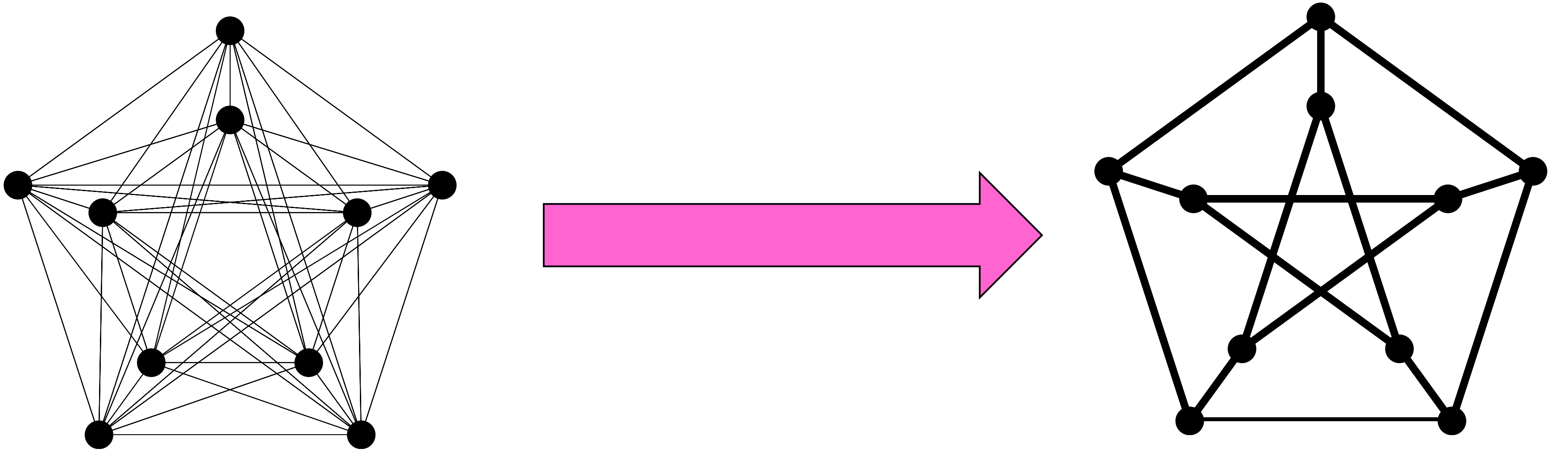
EPFL, June 11, 2012

$$L_G^{-1/2} L_H L_G^{-1/2}$$

This matrix comes up a lot in the talk. It should have a name.

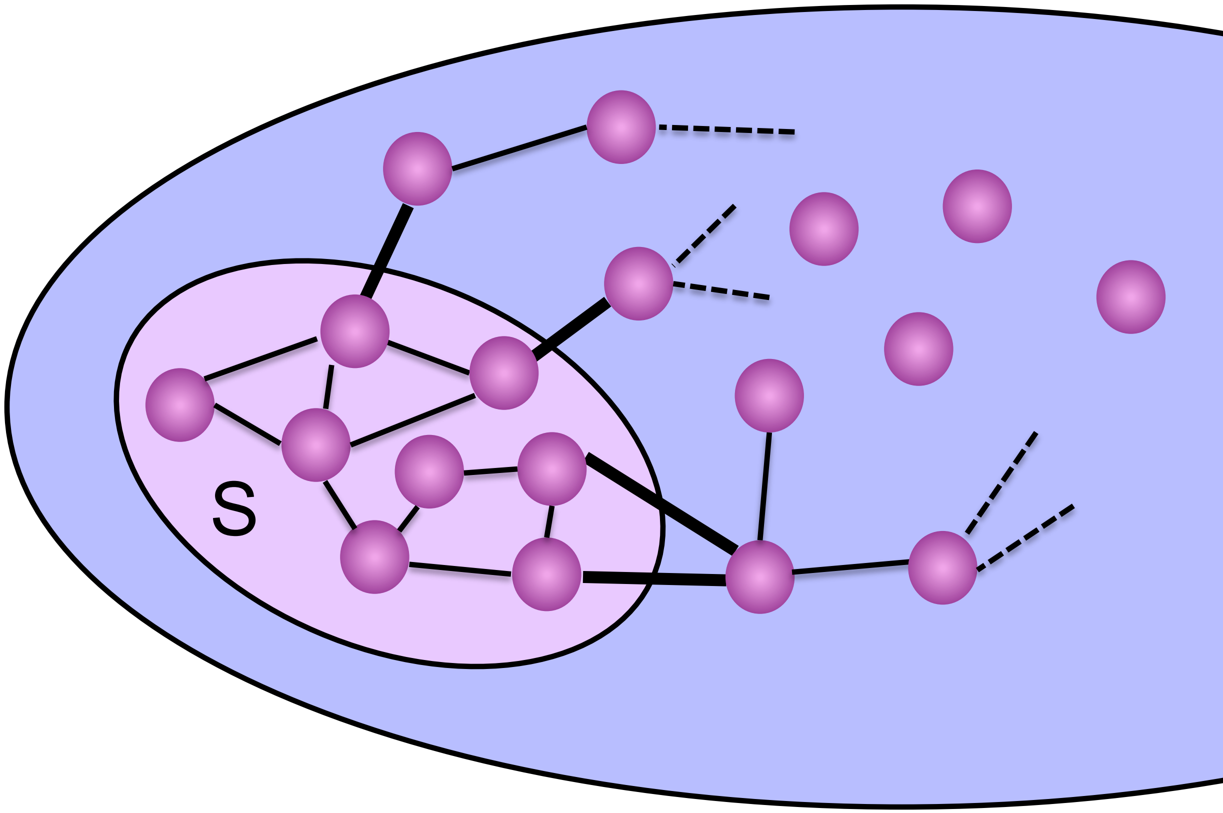
Objective of Sparsification:

Approximate a (weighted) graph by a sparse weighted graph.



Cut Sparsifiers (Benczur-Karger)

Approximate boundaries of sets



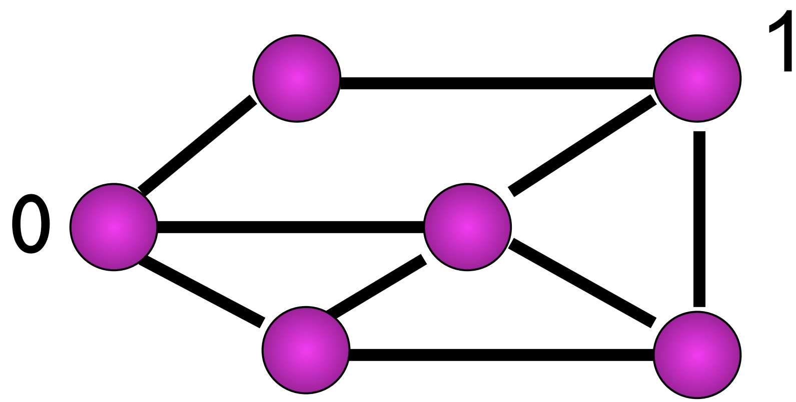
Multiplicative (1+ϵ) approximation with O(n log n / ϵ^2) edges

Learning on Graphs (Zhu-Ghahramani-Lafferty '03)

Infer values of a function at all vertices from known values at a few vertices.

Minimize $$\sum_{(a,b)\in E} (x(a) - x(b))^2$$

Subject to known values



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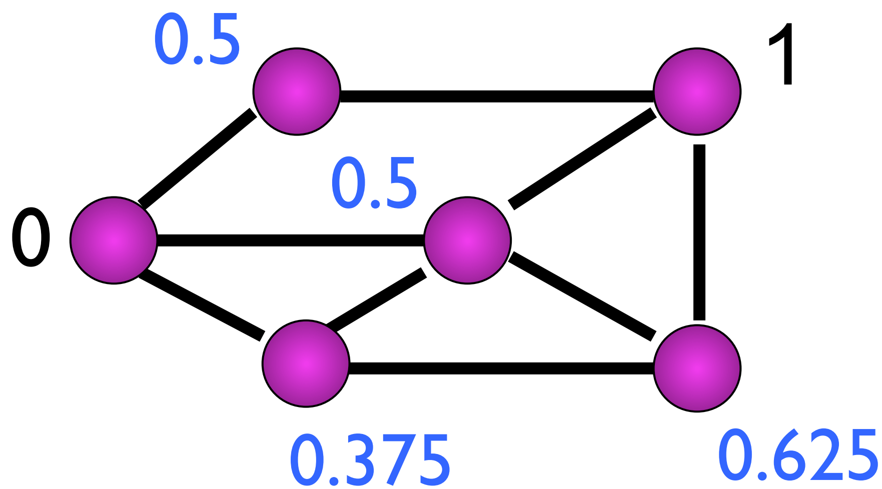
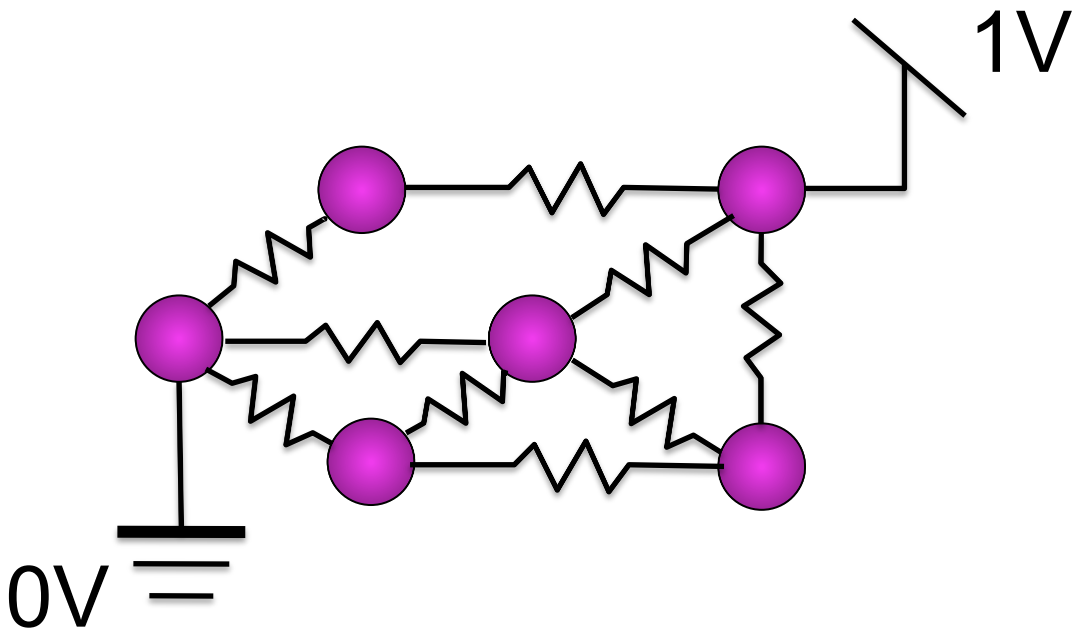


Graphs as Resistor Networks

View edges as resistors connecting vertices

Apply voltages at some vertices.
Measure induced voltages and current flow.



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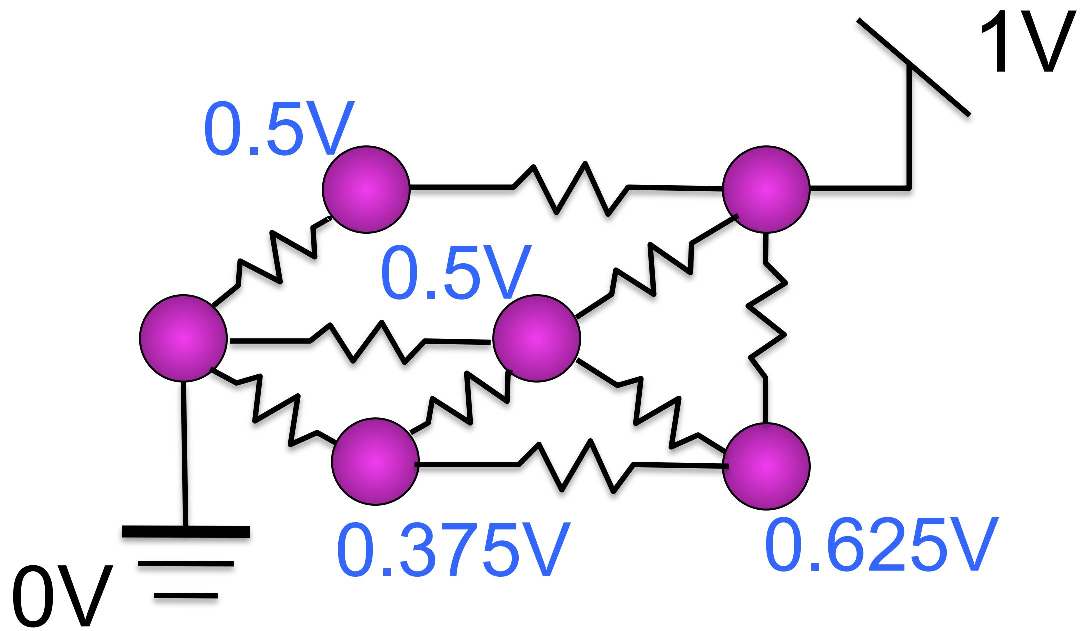


Graphs as Resistor Networks

View edges as resistors connecting vertices

Apply voltages at some vertices.
Measure induced voltages and current flow.

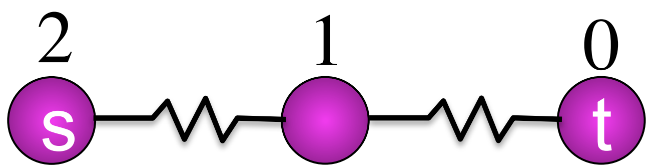
Induced voltages minimize

$$\sum_{(a,b) \in E} (v(a) - v(b))^2$$

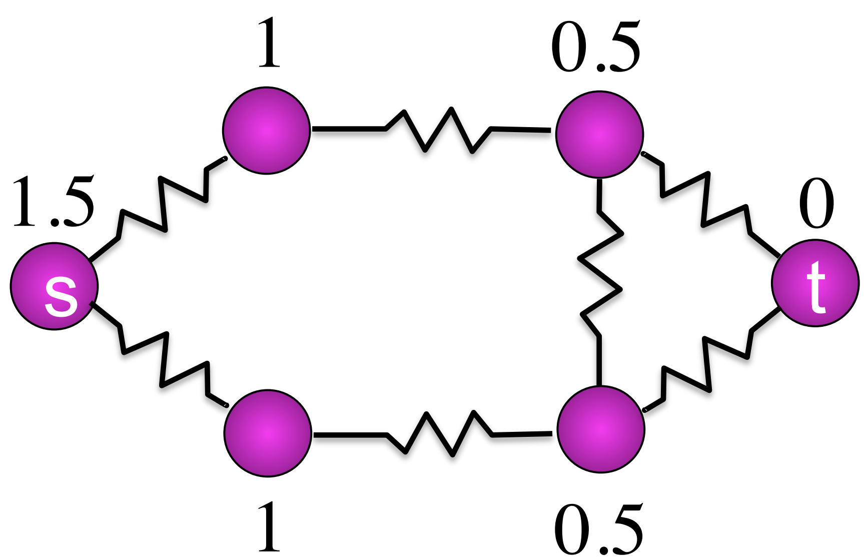
Subject to fixed voltages (by battery)

Graphs as Resistor Networks

Effective Resistance between s and t =
potential difference of unit flow



$$\mathrm{Reff}(s,t) = 2$$



$$\mathrm{Reff}(s,t) = 1.5$$

The Laplacian quadratic form of $G = (V, E)$

$$\sum_{(a,b) \in E} (x(a) - x(b))^2$$

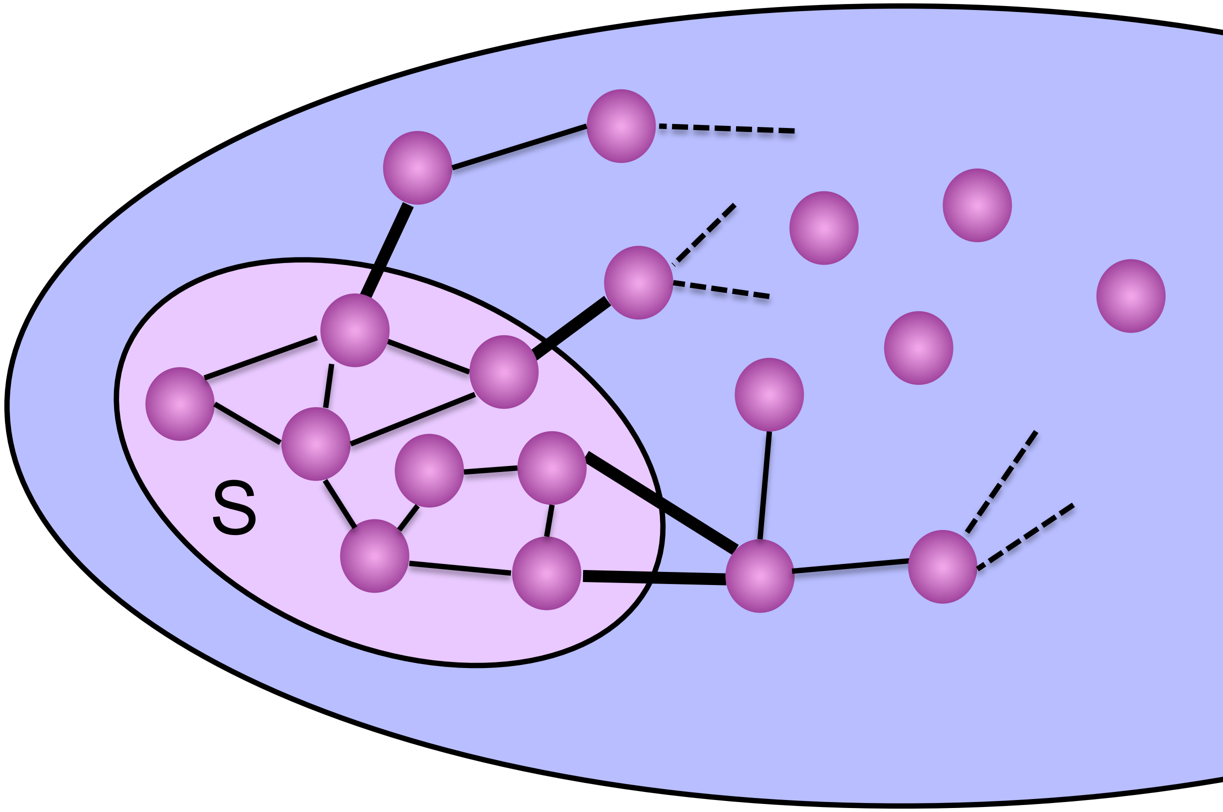
The Laplacian matrix of $G = (V, E)$

$$\sum_{(a,b) \in E} (x(a) - x(b))^2$$

$$= x^T L_G x \qquad x : V \to \mathbb{R}$$

Measuring boundaries of sets

Boundary: edges leaving a set of vertices

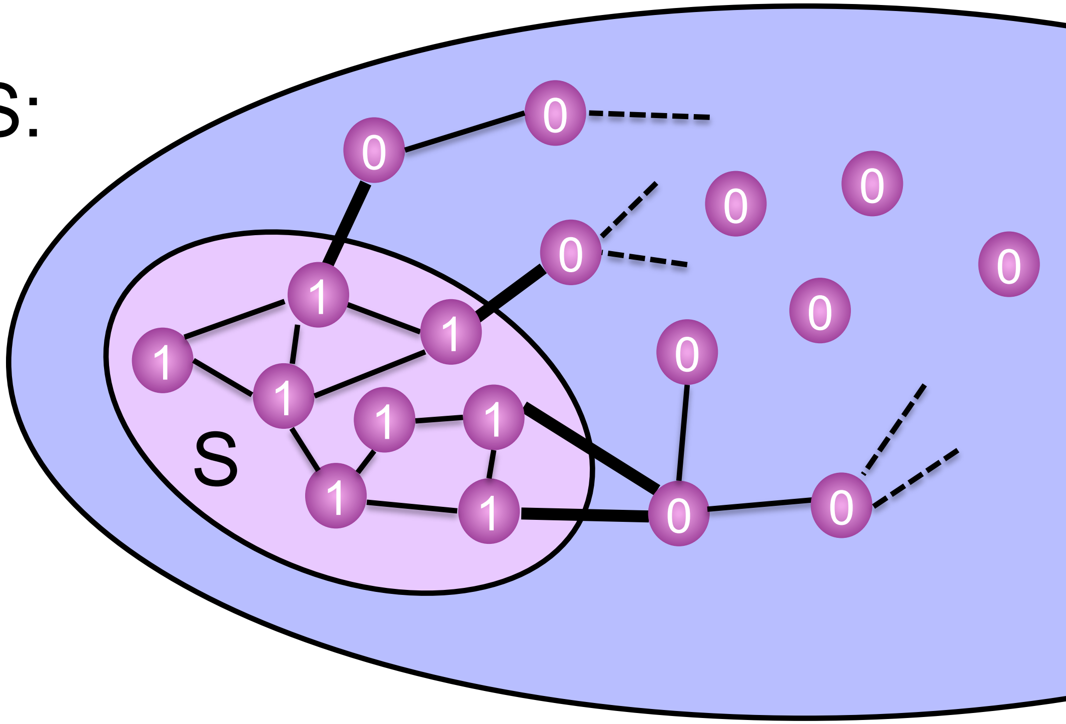


Measuring boundaries of sets

Boundary: edges leaving a set of vertices

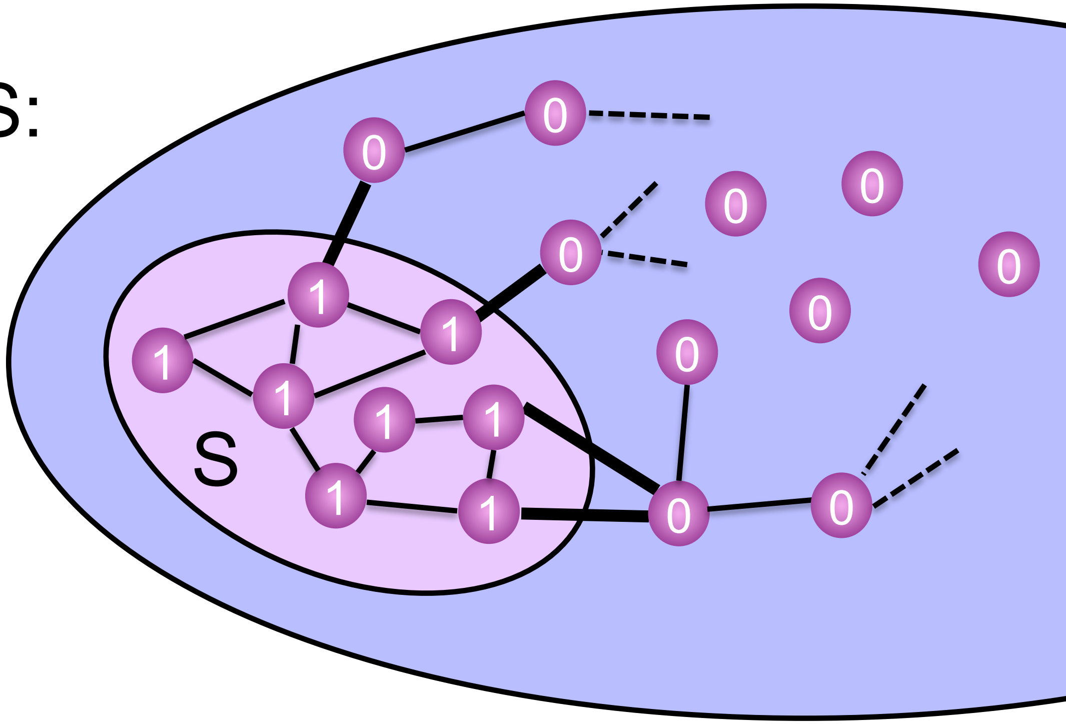
Characteristic Vector of S:

$$x(a) = \begin{cases} 1 & a \text{ in } S \\ 0 & a \text{ not in } S \end{cases}$$



Measuring boundaries of sets

Boundary: edges leaving a set of vertices



Characteristic Vector of S:

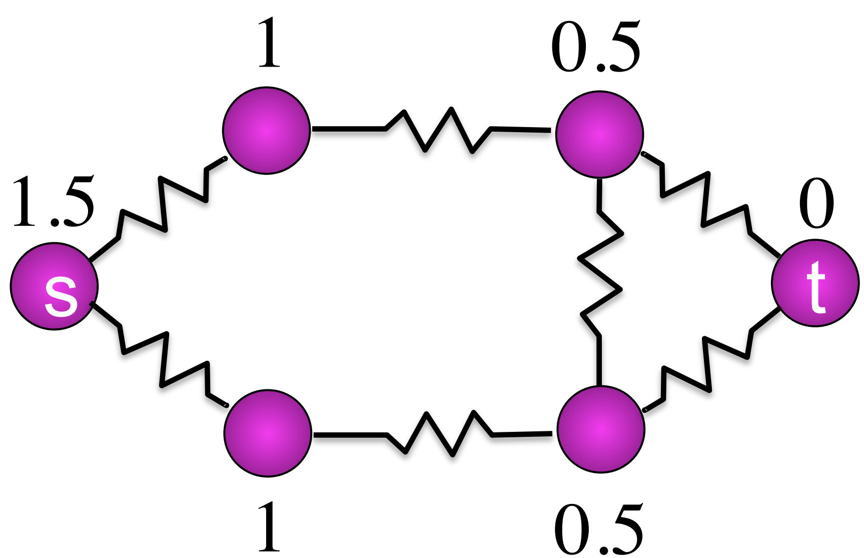
$$x(a) = \begin{cases} 1 & a \text{ in } S \\ 0 & a \text{ not in } S \end{cases}$$

$$x^T L x = \sum_{(a,b) \in E} (x(a) - x(b))^2 = |\text{boundary}(S)|$$

Effective Resistance

$$\mathrm{Reff}(s,t) = (\delta_s - \delta_t)^T L_G^{-1}(\delta_s - \delta_t)$$

Potentials of flow
1 out of s and 1 in to t



$$\mathrm{Reff}(s,t) = 1.5$$

Effective Resistance

$$\mathrm{Reff}(s,t) = (\delta_s - \delta_t)^T \underbrace{L_G^{-1}(\delta_s - \delta_t)}$$

Potentials of flow
1 out of s and 1 in to t

Whenever I write the inverse of a Laplacian, I really mean the pseudo-inverse. As I explain later, we know what the nullspace is. So, it is easy to work orthogonal to the nullspace.

Spectral Sparsification [S-Teng]

Given G, find a sparse graph H for which

$$x^T L_G x \approx x^T L_H x \quad \forall x$$

Spectral Sparsification [S-Teng]

Given G, find a sparse graph H for which

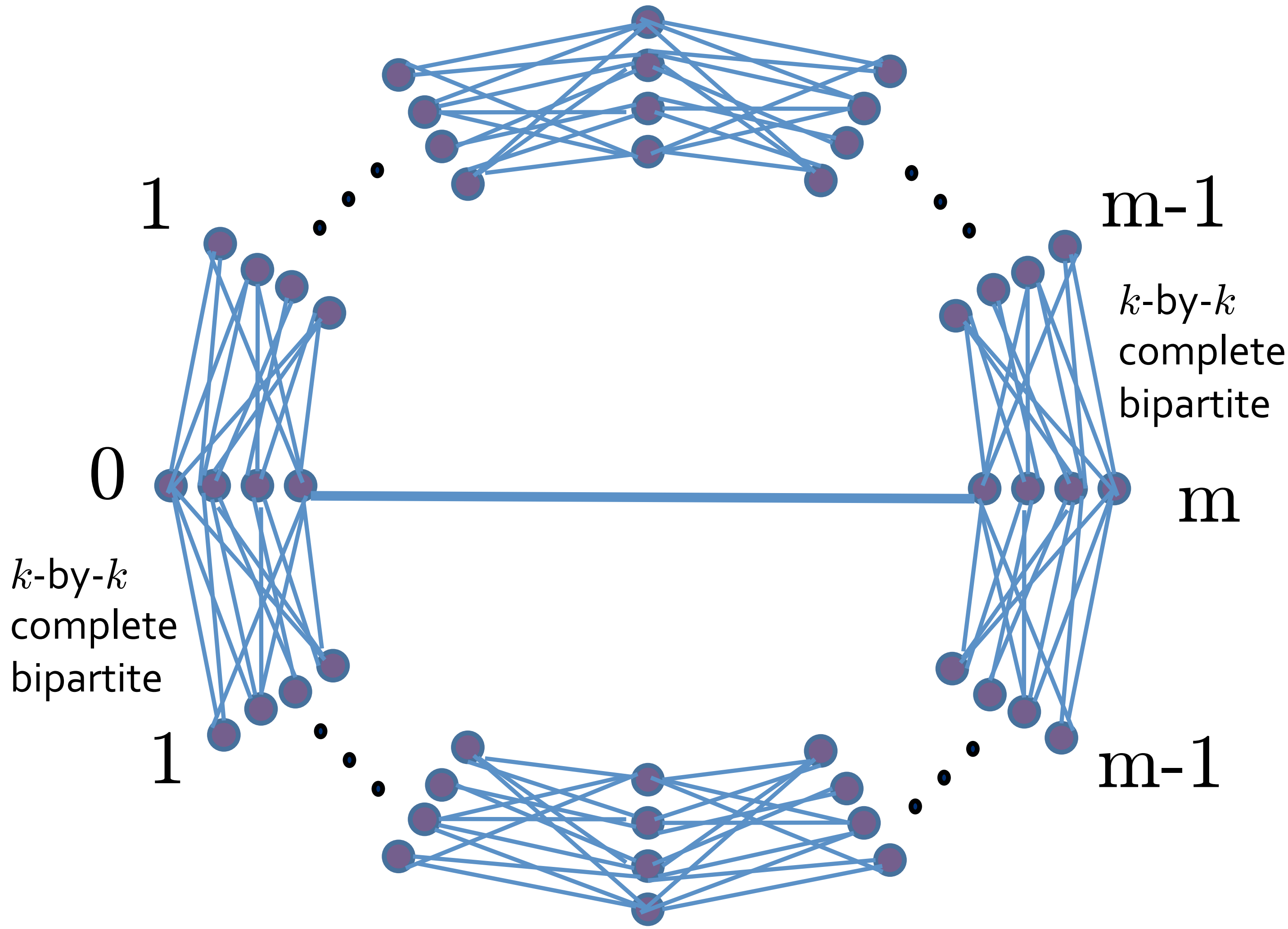
$$x^T L_G x \approx x^T L_H x \quad \forall x$$

In particular, we require

$$\frac{1}{1+\epsilon} \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon$$

In which case we call H an ϵ-approximation of G

Cut-approximation is different



$$x^T L_G x = m^2 + 2mk^2 \qquad (k^2 < m)$$

Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$L_G \preccurlyeq L_H$$

If $L_H - L_G$ is positive semi-definite

Iff, for all $x : V \to \mathbb{R}$

$$x^T L_G x \leq x^T L_H x$$

Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$G \preccurlyeq H$$

If $L_H - L_G$ is positive semi-definite

Iff, for all $x : V \to \mathbb{R}$

$$x^T L_G x \leq x^T L_H x$$

Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$G \preccurlyeq k \cdot H$$ *(multiply edge weights by k)*

If $k \cdot L_H - L_G$ is positive semi-definite

Iff, for all $x : V \to \mathbb{R}$

$$x^T L_G x \leq k \cdot x^T L_H x$$

Approximation

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

H is an ϵ-approximation of G if

$$(1+\epsilon)^{-1} G \preccurlyeq H \preccurlyeq (1+\epsilon) G$$

Approximation

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

H is an ϵ-approximation of G if

$$(1+\epsilon)^{-1} G \preccurlyeq H \preccurlyeq (1+\epsilon) G$$

$$\Updownarrow$$

$$\frac{1}{1+\epsilon} \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon$$

Implications of Approximation

$$(1+\epsilon)^{-1}G \preccurlyeq H \preccurlyeq (1+\epsilon)G$$

Boundaries of sets are similar.

L_H and L_G have similar eigenvalues

$$(1+\epsilon)^{-1}L_H^{-1} \preccurlyeq L_G^{-1} \preccurlyeq (1+\epsilon)L_H^{-1}$$

Effective resistances between vertices similar.

Solutions of linear equations are similar

Implications of Approximation

Solutions of linear equations are similar
(the same holds for regression/learning problems)

$$L_G x = b$$
$$L_H y = b$$

$$\Rightarrow \quad \|x - y\|_{L_G} \leq \epsilon \|x\|_{L_G}$$

$$\|x\|_{L_G} = \sqrt{x^T L_G x} = ||L_G^{1/2} x||$$

Solutions of linear equations are similar

Want to show $\|x - y\|_{L_G} \leq \epsilon \|x\|_{L_G}$

$$L_G x = b$$
$$L_H y = b$$

$\Rightarrow$

$$y = L_H^{-1} L_G x$$

$$\begin{aligned} L_G^{1/2}(x - y) &= L_G^{1/2} x - L_G^{1/2} L_H^{-1} L_G x \\ &= \underline{(I - L_G^{1/2} L_H^{-1} L_G^{1/2})} L_G^{1/2} x \end{aligned}$$

Solutions of linear equations are similar

$$I - L_G^{1/2} L_H^{-1} L_G^{1/2}$$

$$\begin{aligned} L_G^{1/2} L_H^{-1} L_G^{1/2} &\succcurlyeq (1+\epsilon)^{-1} L_G^{1/2} L_G^{-1} L_G^{1/2} \\ &= (1+\epsilon)^{-1} I \\ &\succcurlyeq (1-\epsilon) I \end{aligned}$$

So, $$I - L_G^{1/2} L_H^{-1} L_G^{1/2} \preccurlyeq \epsilon I$$

Solutions of linear equations are similar

$$I - L_G^{1/2} L_H^{-1} L_G^{1/2}$$

So, $$I - L_G^{1/2} L_H^{-1} L_G^{1/2} \preccurlyeq \epsilon I$$

Similarly, $$I - L_G^{1/2} L_H^{-1} L_G^{1/2} \succcurlyeq -\epsilon I$$

$\Updownarrow$

$$\left\| I - L_G^{1/2} L_H^{-1} L_G^{1/2} \right\| \leq \epsilon$$

$\Updownarrow$

$$\left\| (I - L_G^{1/2} L_H^{-1} L_G^{1/2}) x \right\| \leq \epsilon \left\| x \right\|, \ \forall x$$

Solutions of linear equations are similar

$$L_G^{1/2}(x-y) = (I - L_G^{1/2} L_H^{-1} L_G^{1/2}) L_G^{1/2} x$$

$$\left\| (I - L_G^{1/2} L_H^{-1} L_G^{1/2}) x \right\| \leq \epsilon \left\| x \right\|, \ \forall x$$

$$\left\| L_G^{1/2}(x-y) \right\| \leq \epsilon \left\| L_G^{1/2} x \right\|$$

L_H is a good preconditioner for L_G

$$\left\| L_G^{1/2}(x-y) \right\| \leq \epsilon \left\| L_G^{1/2} x \right\|$$

By solving equation in residual,
can solve equations in L_G by solving in L_H

Laplacian solvers of (S-Teng, Koutis-Miller-Peng)
provide similar guarantees,
but L_H^{-1} is implicit, and L_H is never constructed

Main Theorems

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

1. H is an ϵ-approximation of G
2. $|F| \leq |V| (2+\epsilon)^2/\epsilon^2$ (Batson-S-Srivastava 09)
3. $F \subseteq E$

By careful random sampling, get

$|F| \leq O(|V| \log |V| / \epsilon^2)$ (S-Srivastava 08)

Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v)\in E} w_{u,v}(x(u) - x(v))^2$$

$$L_G = D_G - A_G$$

Positive semi-definite

If connected, nullspace = Span($\mathbf{1}$)

As understand nullspace, can pretend invertible

Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v)\in E} w_{u,v}(x(u) - x(v))^2$$

$$L_G = \sum_{(u,v)\in E} w_{u,v} L_{u,v}$$

E.g. $$L_{1,2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}$$

Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2$$

$$L_G = \sum_{(u,v) \in E} w_{u,v} L_{u,v}$$

$$= \sum_{(u,v) \in E} w_{u,v} (b_{u,v} b_{u,v}^T)$$

where $b_{u,v} = \delta_u - \delta_v$

Sum of outer products

Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v) \in E} w_{u,v}(x(u) - x(v))^2$$

$$L_G = \sum_{(u,v) \in E} w_{u,v}(b_{u,v} b_{u,v}^T)$$

$$= BWB^T$$

m-by-m diagonal

n-by-m matrix with columns $b_{u,v}$

Sparsification by Random Sampling

Assign a probability $p_{u,v}$ to each edge (u,v)

Include edge (u,v) in H with probability $p_{u,v}$.

If include edge (u,v), give it weight $w_{u,v}/p_{u,v}$

$$\mathbb{E}\left[\ L_H\ \right] = \sum_{(u,v)\in E} p_{u,v}(w_{u,v}/p_{u,v})L_{u,v} = L_G$$

Sparsification by Random Sampling

Choose $p_{u,v}$ to be $w_{u,v}$ times the effective resistance between u and v.

Low resistance between u and v means there are many alternate routes for current to flow and that the edge is not critical.

$$\mathrm{Reff}(u, v) = b_{u,v}^T L_G^{-1} b_{u,v}$$

Simplification of Sparsification

$$L_H \preccurlyeq (1+\epsilon) L_G$$

$$\Updownarrow$$

$$M^T L_H M \preccurlyeq (1+\epsilon) M^T L_G M$$

Using $M = L_G^{-1/2}$ $\Updownarrow$

$$L_G^{-1/2} L_H L_G^{-1/2} \preccurlyeq (1+\epsilon) I$$

Simplification of Sparsification

$$(1-\epsilon)L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon)L_G$$

$\Updownarrow$

$$(1-\epsilon)I \preccurlyeq L_G^{-1/2} L_H L_G^{-1/2} \preccurlyeq (1+\epsilon)I$$

$\Updownarrow$

$$\left\| I - L_G^{-1/2} L_H L_G^{-1/2} \right\| \leq \epsilon$$

Analysis of Random Sampling

$$\left\| I - L_G^{-1/2} L_H L_G^{-1/2} \right\| \leq \epsilon$$

$$\mathbb{E}\left[\; L_H \;\right] = L_G, \text{ so } \mathbb{E}\left[\; L_G^{-1/2} L_H L_G^{-1/2} \;\right] = I$$

Analysis of Random Sampling

$$\left\| I - L_G^{-1/2} L_H L_G^{-1/2} \right\| \leq \epsilon$$

$$\mathbb{E}\left[\ L_H\ \right] = L_G, \text{ so } \mathbb{E}\left[\ L_G^{-1/2} L_H L_G^{-1/2}\ \right] = I$$

$$L_G = \sum_{(u,v)\in E} w_{u,v} b_{u,v} b_{u,v}^T$$

$$\Downarrow$$

$$L_G^{-1/2} L_G L_G^{-1/2} = \sum_{(u,v)\in E} w_{u,v} \left(L_G^{-1/2} b_{u,v} \right) \left(b_{u,v}^T L_G^{-1/2} \right)$$

Rudelson's Concentration Theorem ('99)

If $y_1, \ldots, y_m$ are i.i.d. random vectors s.t.

$$\mathbb{E}\, y_i y_i^T = I \quad \text{and} \quad \|y_i\| \leq t \text{ a.s.}$$

Then

$$\mathbb{E}\left[\left\| I - \frac{1}{m}\sum_i y_i y_i^T \right\|_2\right] \leq \mathrm{const} \cdot t\sqrt{\frac{\log m}{m}}$$

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Then

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Applying Rudelson's Concentration

$$L_G^{-1/2} L_G L_G^{-1/2} = \sum_{(u,v)\in E} w_{u,v} \left(L_G^{-1/2} b_{u,v}\right)\left(b_{u,v}^T L_G^{-1/2}\right)$$

$$= \sum_{(u,v)\in E} p_{u,v} \frac{w_{u,v} \left(L_G^{-1/2} b_{u,v}\right)\left(b_{u,v}^T L_G^{-1/2}\right)}{p_{u,v}}$$

Applying Rudelson's Concentration

$$= \sum_{(u,v)\in E} p_{u,v} \frac{w_{u,v}\left(L_G^{-1/2} b_{u,v}\right)\left(b_{u,v}^T L_G^{-1/2}\right)}{p_{u,v}}$$

To make all of these have the same norm,

we set

$$p_{u,v} = w_{u,v} \left\| \left(L_G^{-1/2} b_{u,v}\right)\left(b_{u,v}^T L_G^{-1/2}\right) \right\|_2$$

$$= w_{u,v} b_{u,v}^T L_G^{-1} b_{u,v}$$

Sparsification by Random Sampling

For every $G = (V, E, w)$, can find an $H = (V, F, z)$ s.t.

1. H is an ϵ-approximation of G

2. H has $O(|V| \log |V| / \epsilon^2)$ edges

3. In time $O(|E| \log^2 |V| \log(1/\epsilon))$

[Koutis-Levin-Peng '12, using S-Teng, Koutis-Miller-Peng]

Quickly computing effective resistances

Need to compute $b_{u,v}^T L_G^{-1} b_{u,v}$ for all edges (u,v)

Idea 1: $b_{u,v}^T L_G^{-1} b_{u,v} = ||L_G^{-1/2} b_{u,v}||^2$

Norms are preserved under random projection (Johnson-Lindenstrauss)

$$\approx ||R L_G^{-1/2} b_{u,v}||^2$$

R an O(log n)-by-n dimensional matrix

Quickly computing effective resistances

Need to compute $b_{u,v}^T L_G^{-1} b_{u,v}$ for all edges (u,v)

Idea 1: $b_{u,v}^T L_G^{-1} b_{u,v} = ||L_G^{-1/2} b_{u,v}||^2$

This idea doesn't work because I don't know how quickly apply the square root of the inverse of L_G. But, we don't really need this square root. Any matrix M for which $M^T M$ equals the inverse of the Laplacian will do.

Quickly computing effective resistances

Idea 2: $$\begin{aligned} L_G^{-1} &= L_G^{-1} L_G L_G^{-1} \\ &= L_G^{-1} B W B^T L_G^{-1} \\ &= (L_G^{-1} B W^{1/2})(W^{1/2} B^T L_G^{-1}) \end{aligned}$$

$$b_{u,v} L_G^{-1} b_{u,v} = ||W^{1/2} B^T L_G^{-1} b_{u,v}||^2$$

$$b_{u,v} L_G^{-1} b_{u,v} \approx ||R W^{1/2} B^T L_G^{-1} b_{u,v}||^2$$

R a random O(log n)-by-m dimensional matrix

Quickly computing effective resistances

$$b_{u,v} L_G^{-1} b_{u,v} \approx ||RW^{1/2} B^T L_G^{-1} b_{u,v}||^2$$

R a random O(log n)-by-m dimensional matrix

Can ϵ-approximate rows of $RW^{1/2} B^T L_G^{-1}$

in time $\widetilde{O}(m \log^2 n \log \epsilon^{-1})$

using fast Laplacian solver

(S-Teng, Koutis-Miller-Peng)

Quickly computing effective resistances

$$b_{u,v} L_G^{-1} b_{u,v} \approx ||R W^{1/2} B^T L_G^{-1} b_{u,v}||^2$$

R a random O(log n)-by-m dimensional matrix

Can ϵ-approximate rows of $R W^{1/2} B^T L_G^{-1}$

in time $\widetilde{O}(m \log^2 n \log \epsilon^{-1})$

$$b_{u,v} = \delta_u - \delta_v$$

So, each Reff is difference of two rows

Quickly computing effective resistances

Srivastava and I didn't take advantage of the fact that the fast solvers are linear operators, and our analysis tried to get the error much lower in the 2-norm.

Koutis, Levin and Peng get a much simpler and tighter analysis by exploiting the facts that the fast solvers are linear operators and that they give small error in the matrix norm.

I give the analysis in one slide.

Faster computation of effective resistances

(Koutis-Levin-Peng)

Fast Laplacian solvers:

1. find approximations in the matrix norm
2. are linear operators

That is, exists L_F so that

1. L_F^{-1} can be computed quickly, and
2. F is an ϵ-approximation of G

Faster computation of effective resistances

(Koutis-Levin-Peng)

$$b_{u,v}^T L_G^{-1} b_{u,v} \leq (1+\epsilon) b_{u,v}^T L_F^{-1} b_{u,v}$$

Faster computation of effective resistances

(Koutis-Levin-Peng)

$$\begin{aligned} b_{u,v}^T L_G^{-1} b_{u,v} &\leq (1+\epsilon) b_{u,v}^T L_F^{-1} b_{u,v} \\ &= (1+\epsilon) b_{u,v}^T L_F^{-1} L_F L_F^{-1} b_{u,v} \end{aligned}$$

Faster computation of effective resistances

(Koutis-Levin-Peng)

$$b_{u,v}^T L_G^{-1} b_{u,v} \leq (1+\epsilon) b_{u,v}^T L_F^{-1} b_{u,v}$$
$$= (1+\epsilon) b_{u,v}^T L_F^{-1} L_F L_F^{-1} b_{u,v}$$

L_F is never constructed, so use L_G instead

$$\leq (1+\epsilon)^2 b_{u,v}^T L_F^{-1} L_G L_F^{-1} b_{u,v}$$
$$= (1+\epsilon)^2 b_{u,v}^T L_F^{-1} B W B^T L_F^{-1} b_{u,v}$$

So, can get constant approx from constant ϵ

Sparsification by Careful Construction

For every $G = (V, E, w)$, can find an $H = (V, F, z)$ s.t.

1. H is an ϵ-approximation of G

2. H has at most $n(2+\epsilon)^2/\epsilon^2$ edges

3. In polynomial time.

Follows from improved, deterministic, variant of Rudelson's theorem.

Very Sparse Approximations

1. H is an ϵ-approximation of G

2. H has at most $n(2+\epsilon)^2/\epsilon^2$ edges

For big ϵ, $\quad (2+\epsilon)^2/\epsilon^2 \approx (1+4/\epsilon)$

Get n plus a small number of edges

Very Sparse Approximations

1. H is a k^2-aproximation of G

2. H has at most $n + O(n/k)$ edges

For bigger k, can do even better:

k-approximation

With $n + \widetilde{O}(n(\log n)/k)$ edges

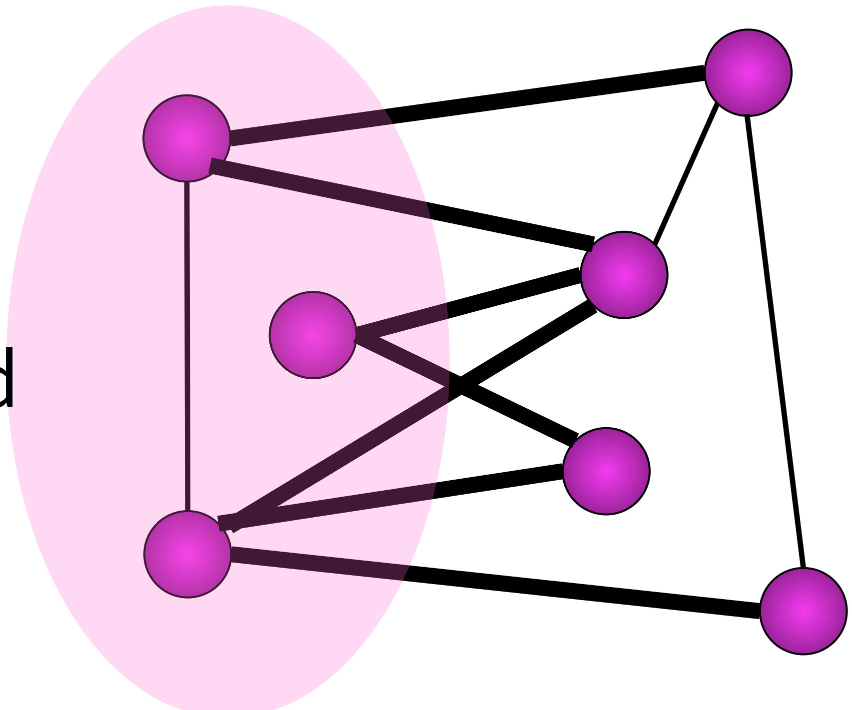
(Kolla-Makarychev-Saberi-Teng)

Uses low-stretch trees, special property of graphs

Max-Cut

Problem: find S maximizing |bdry(S)|

1.13-approx in PTime
(Goemans-Williamson)



(17/16-ϵ)-approx is NP-Hard
(Håstad)

(1.13-ϵ)-approx would
break Unique Games Conj.
(Khot-Kindler-Mossel-O'Donnell)

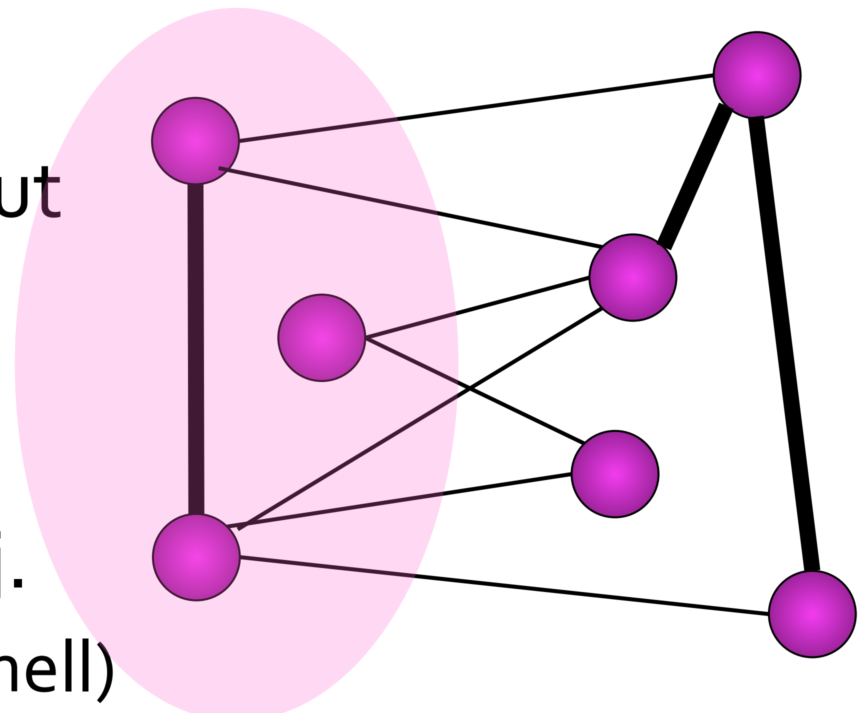
Can sparsify the graph

Min-Uncut

Problem: find S minimizing $|\{(u,v) \notin \mathrm{bdry} S\}|$

If can leave ϵm edges uncut,
can find S with $O(\sqrt{\epsilon} m)$ uncut
(Goemans-Williamson)

Much better would
break Unique Games Conj.
(Khot-Kindler-Mossel-O'Donnell)



Need $\sqrt{\epsilon}$-approximations, which have O(n/ϵ) edges

Min-Uncut

Sparsifying the Laplacian doesn't give very good results for Min-Uncut. But, this problem isn't really about the Laplacian.

In the next slide, we present the sum of rank-1 matrices that we really care about.

Note that we are now getting weighted edges.

Min-Uncut

Problem: find S minimizing $|\{(u,v) \notin \mathrm{bdry} S\}|$

$$\min_{x \in \{\pm 1\}^n} \sum_{(i,j) \in E} (x_i + x_j)^2$$

$$(x_i + x_j)^2 = \begin{pmatrix} x_i \\ x_j \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ x_j \end{pmatrix}$$

Min-Uncut

Problem: find S minimizing $|\{(u,v) \notin \mathrm{bdry} S\}|$

$$\min_{x\in\{\pm 1\}^n} \sum_{(i,j)\in E} (x_i + x_j)^2$$

$$= \min_{x\in\{\pm 1\}^n} \sum_{(i,j)\in E} x^T M_{i,j} x$$

= a sum of rank-1 matrices

Min-Uncut

Problem: find S minimizing $|\{(u,v) \notin \mathrm{bdry} S\}|$

$$\min_{x \in \{\pm 1\}^n} \sum_{(i,j) \in E} (x_i + x_j)^2$$

$$= \min_{x \in \{\pm 1\}^n} \sum_{(i,j) \in E} x^T M_{i,j} x$$

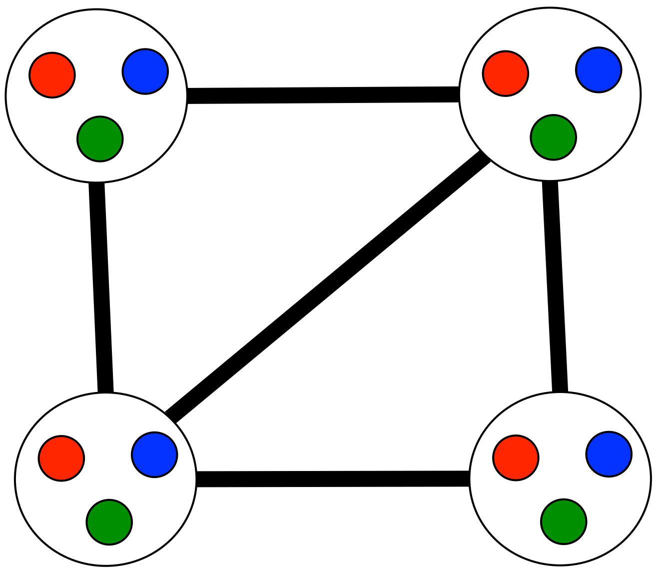
= a sum of rank-1 matrices

Can sparsify, and even make very sparse.
Is it easier with $n + O(n\epsilon^{1/4})$ edges?

Unique Games (Khot)

Graph with n vertices, each given one of k colors.
Each edge (u,v) has permutation $\pi_{u,v}$ on {1,...,k}

Edge (u,v) satisfied if $c_u = \pi_{u,v}(c_v)$



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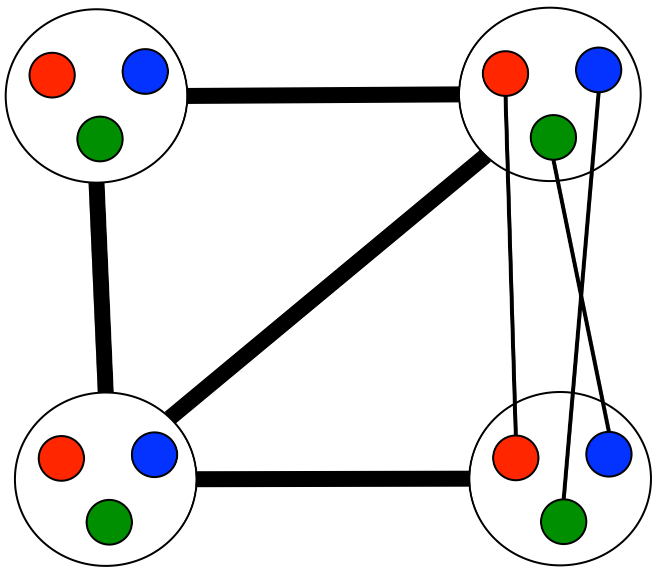
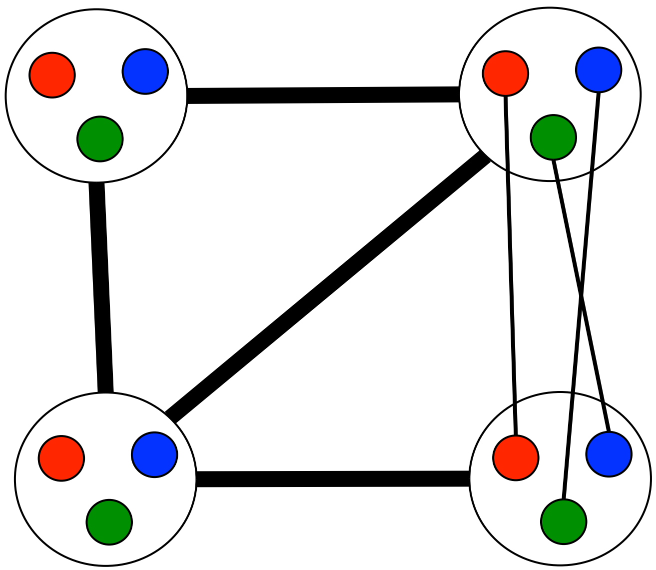


Unique Games Conjecture (Khot)

For every ϵ > 0, there is a k for which
it is hard to satisfy more than ϵn edges,
even if it is possible to satisfy (1-ϵ)n



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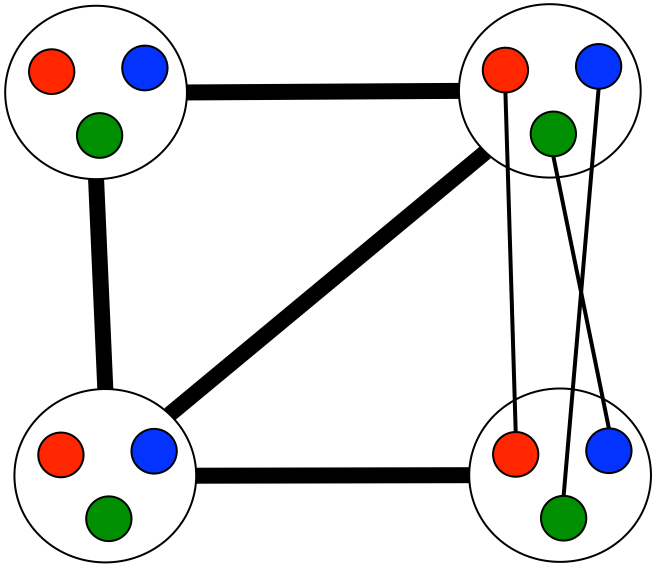

We can sparsify these problems too!

Sparsifying Sums of PSD Matrices

(de Carli Silva, Harvey, Sato)

1. Can generalize Rudelson's theorem to
   sums of PSD matrices.
   Prove via matrix Chernoff bounds
   (Ahlswede and Winter '02)

2. Can generalize BSS sparsification to
   sums of PSD matrices.
   Essentially the same proof.

Sparsifying Unique Games

Map colors {1,...,k} to vectors $\delta_1, \ldots, \delta_k \in \mathbb{R}^k$

Map permutations $\pi_{u,v}$ to matrices $\Pi_{u,v}$

Let each vertex have a vector x_u

Edge (u,v) satisfied if $\|x_u - \Pi_{u,v} x_v\|^2 = 0$

$$\min_{x_u \in \{\delta_1, \ldots, \delta_k\}} \sum_{(u,v) \in E} \|x_u - \Pi_{u,v} x_v\|^2$$

Sparsifying Unique Games

$$\min_{x_u \in \{\delta_1, \ldots, \delta_k\}} \sum_{(u,v) \in E} \left\| x_u - \Pi_{u,v} x_v \right\|^2$$

Sum of PSD matrices in kn dimensions,
so can sparsify to O(kn) weighted edges.

Does this make the problem easier?

Note that this sparsification puts weights on the edges.

Open Questions

Better sparsification of unique game problems?

Application of sparsifying unique games?

Better k-approximations for $k > 1$?

Lower bounds on sparsification?

Fast construction of linear-sized sparsifiers