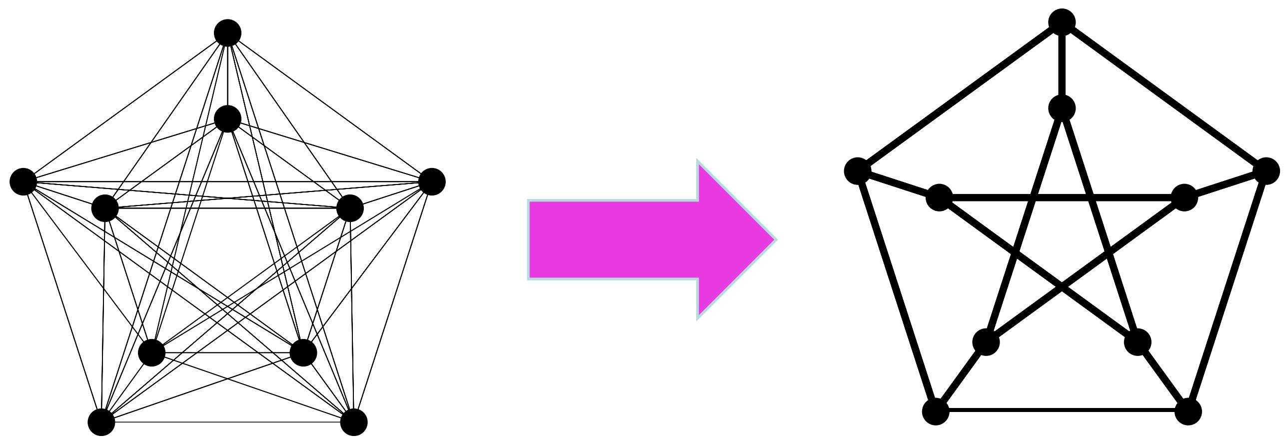
The Laplacian Matrices of Graphs



Daniel A. Spielman

YALE INSTITUTE FOR
NETWORK SCIENCE

ISIT, July 12, 2016

Outline

- Laplacians
  - Interpolation on graphs
  - Resistor networks
  - Spring networks
  - Graph drawing
  - Clustering
  - Linear programming
- Sparsification
- Solving Laplacian Equations
  - Best results
  - The simplest algorithm

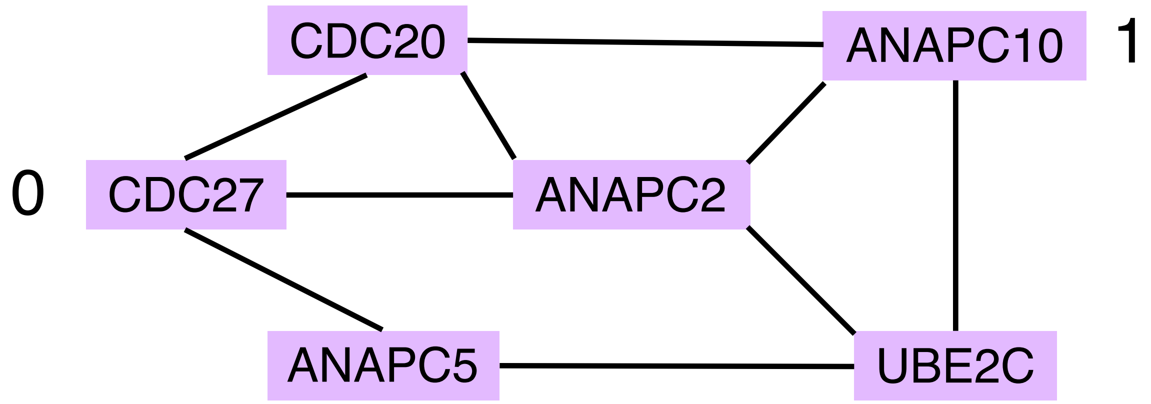
Interpolation on Graphs

(Zhu,Ghahramani,Lafferty '03)

Interpolate values of a function at all vertices from given values at a few vertices.

Minimize $$\sum_{(a,b) \in E} (x(a) - x(b))^2$$

Subject to given values



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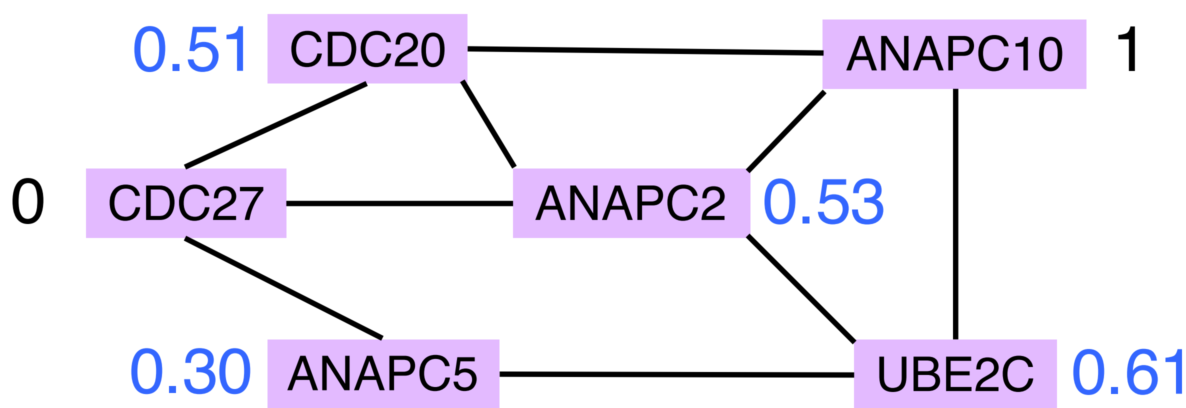

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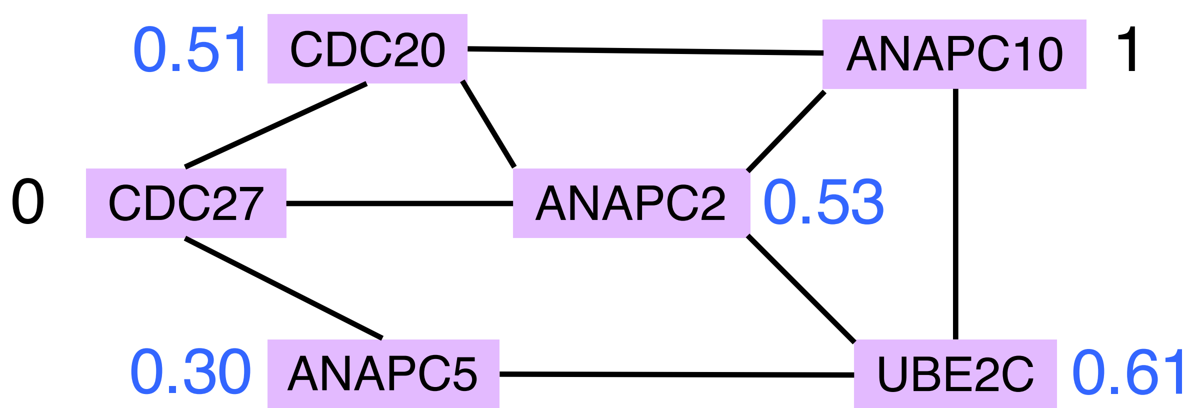

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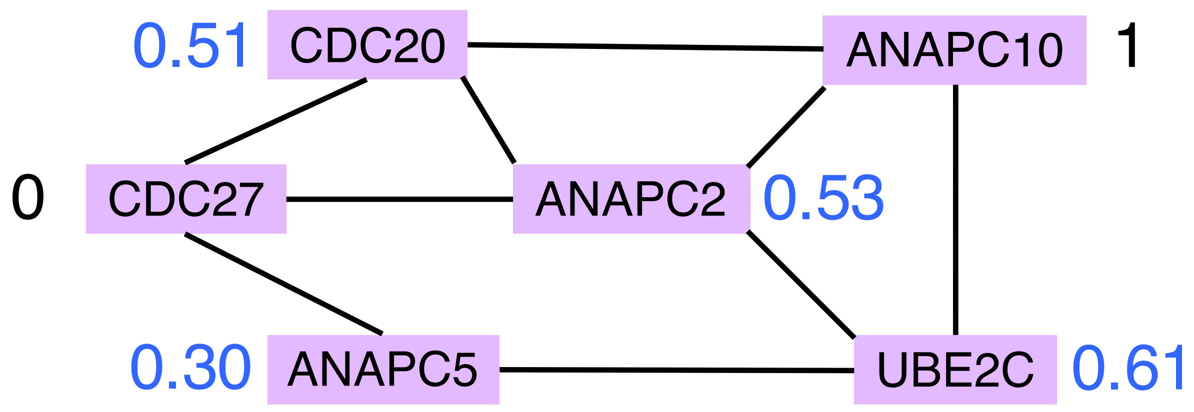
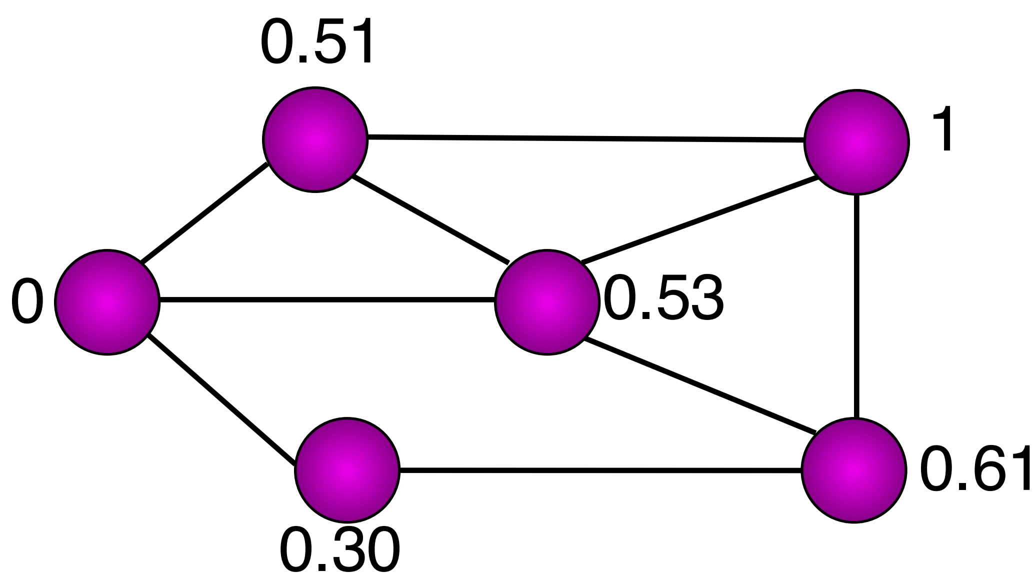


Take derivatives. Minimize by solving Laplacian

The Laplacian Quadratic Form of $G = (V, E)$

$$x : V \to \mathbb{R} \qquad \sum_{(a,b) \in E} (x(a) - x(b))^2$$



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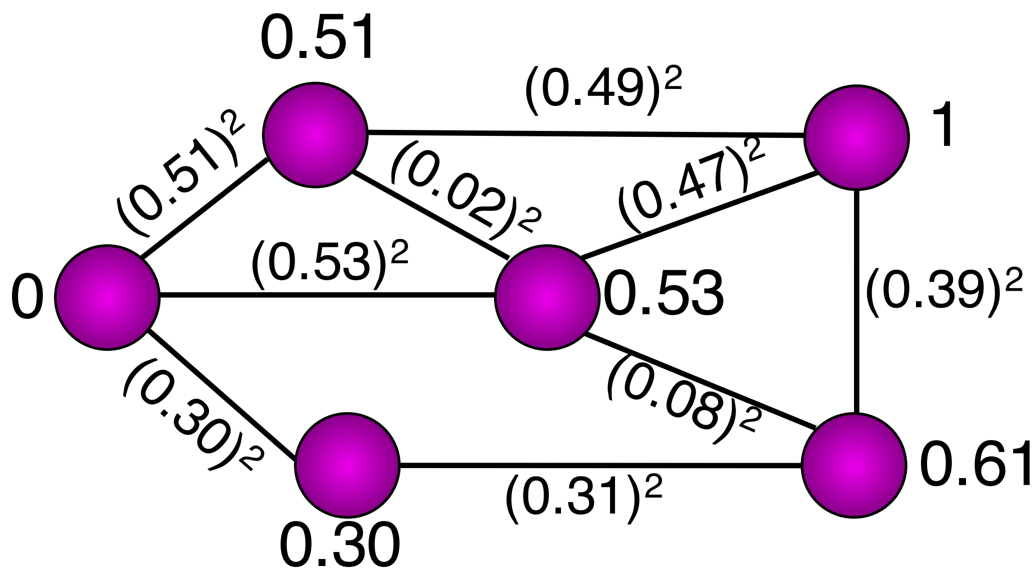


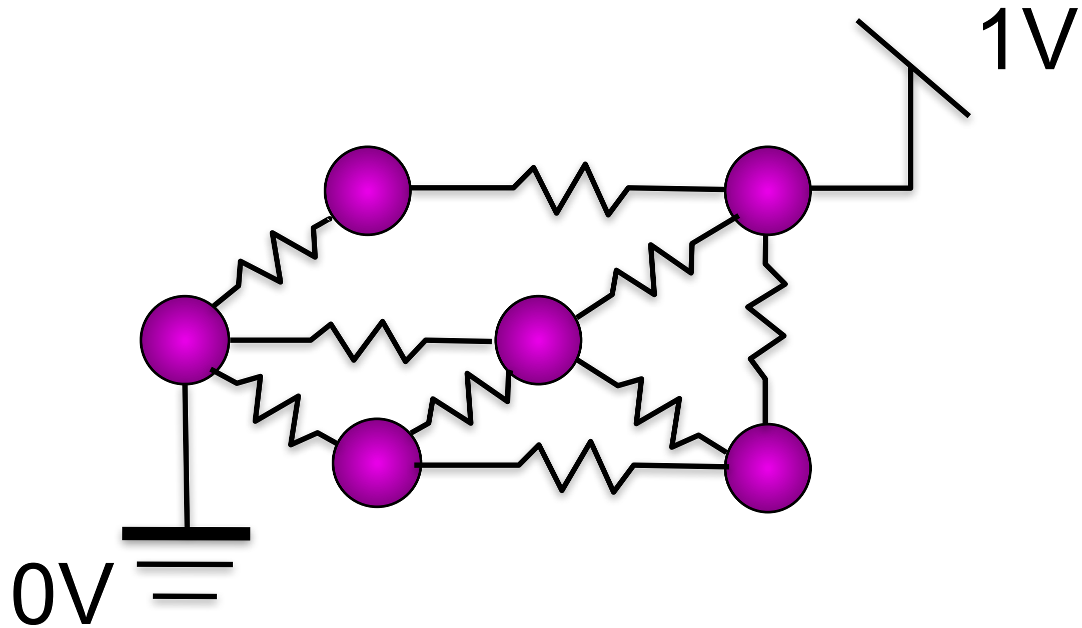
Graphs with positive edge weights

$$\sum_{(a,b)\in E} w_{a,b}(x(a) - x(b))^2 = x^T L_G x$$

Resistor Networks

View edges as resistors connecting vertices

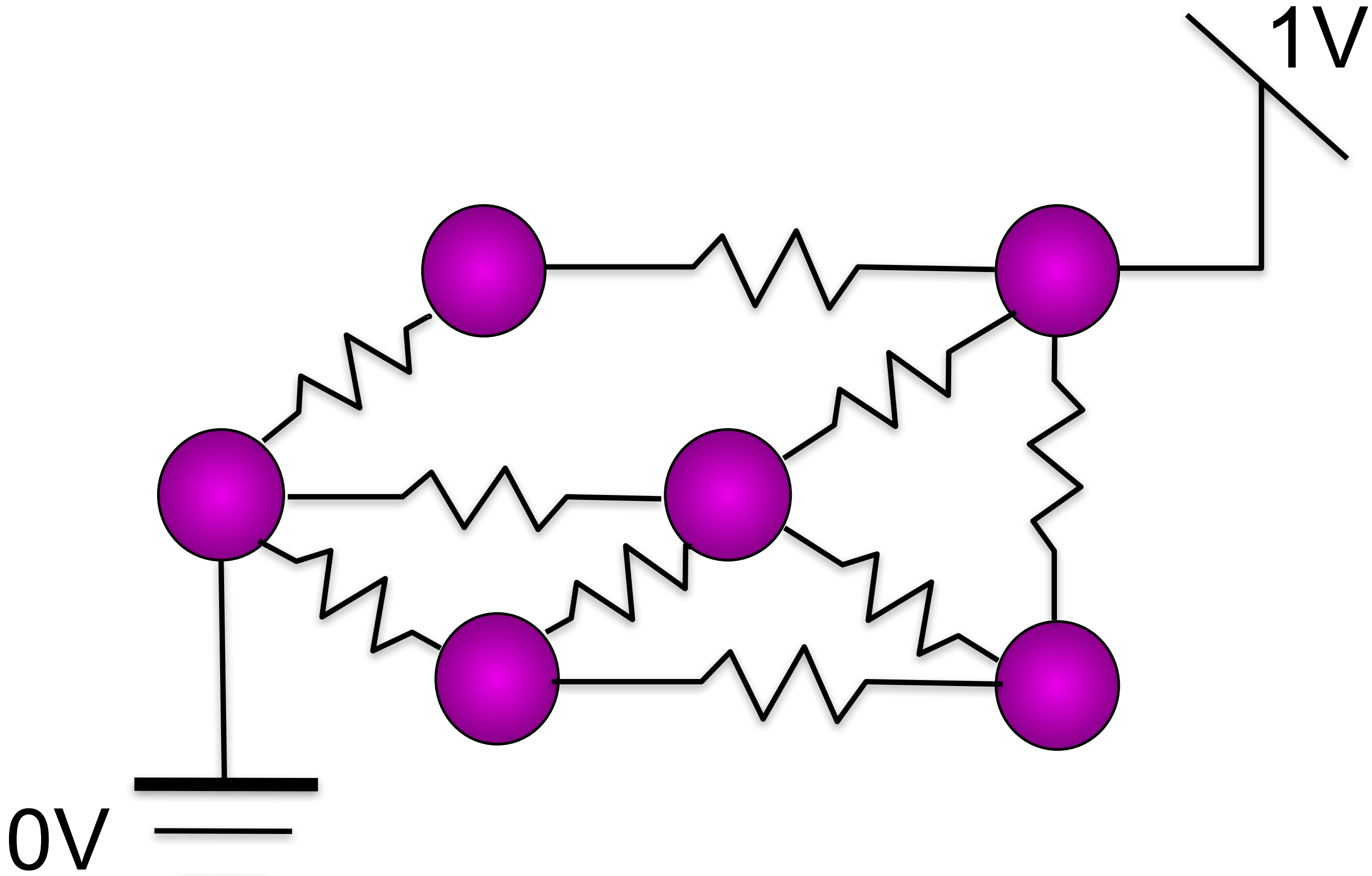
Apply voltages at some vertices.
Measure induced voltages and current flow.



Resistor Networks

Induced voltages minimize subject to constraints.

$$\sum_{(a,b)\in E} (x(a) - x(b))^2$$



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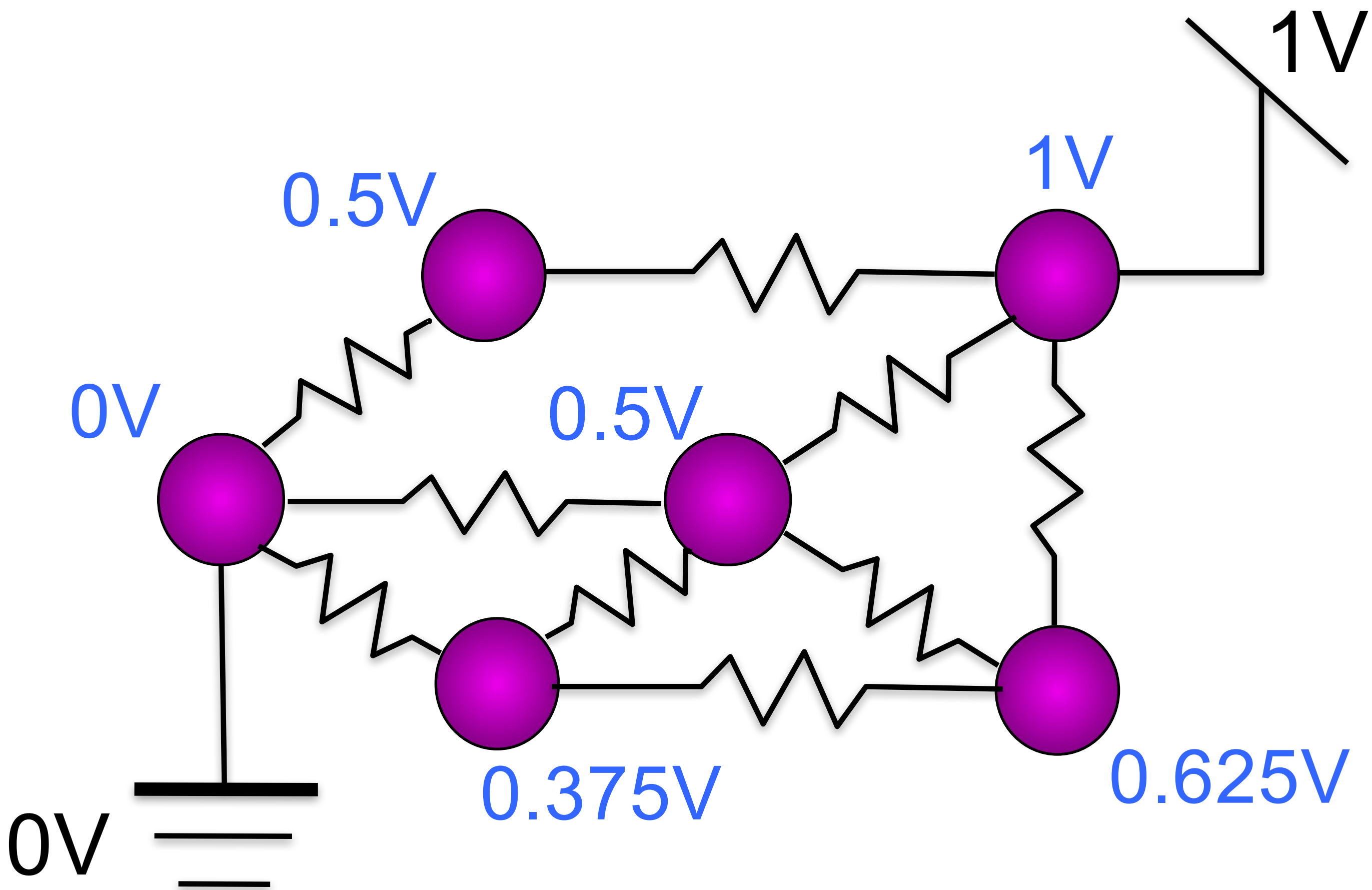

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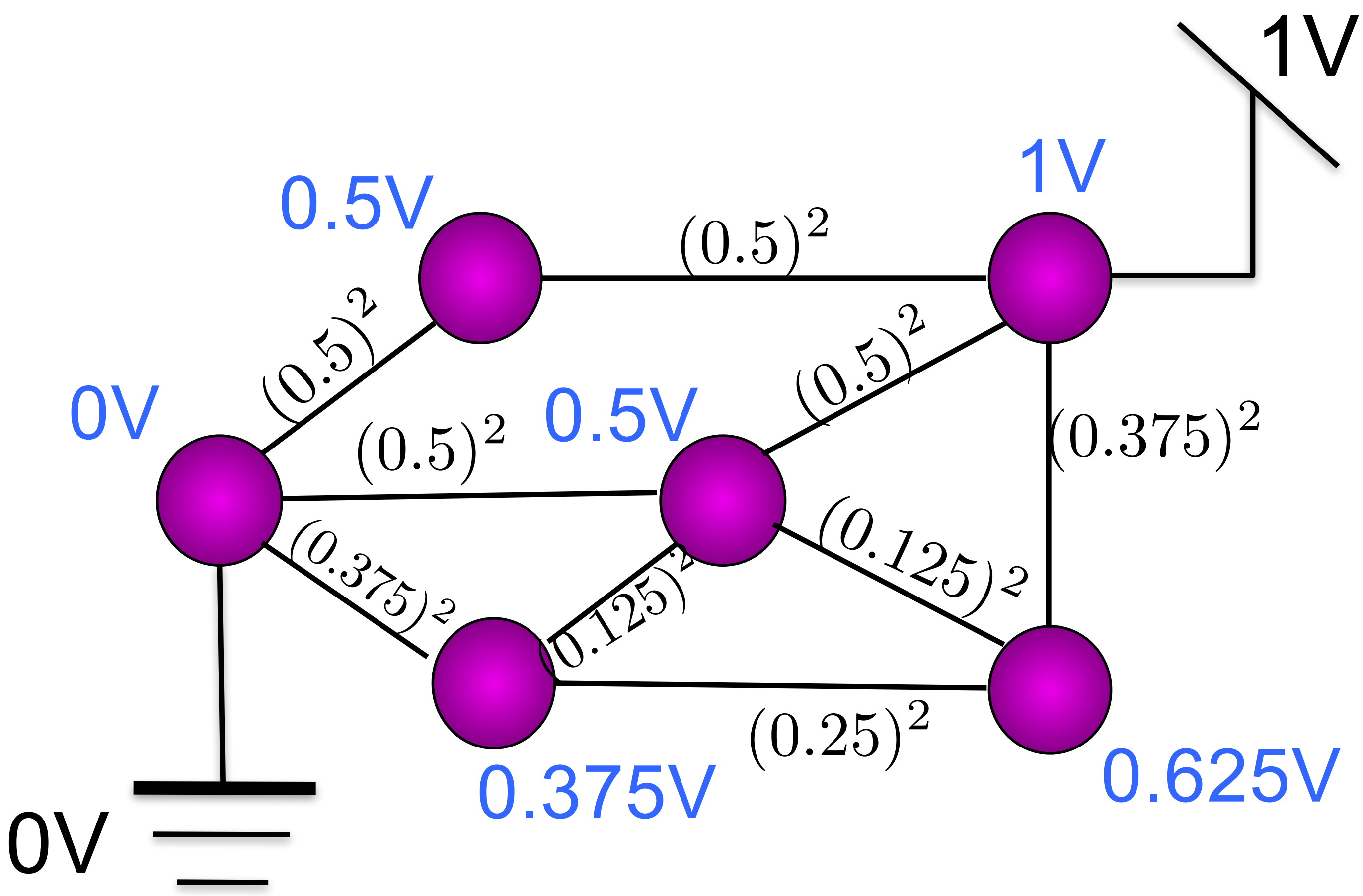
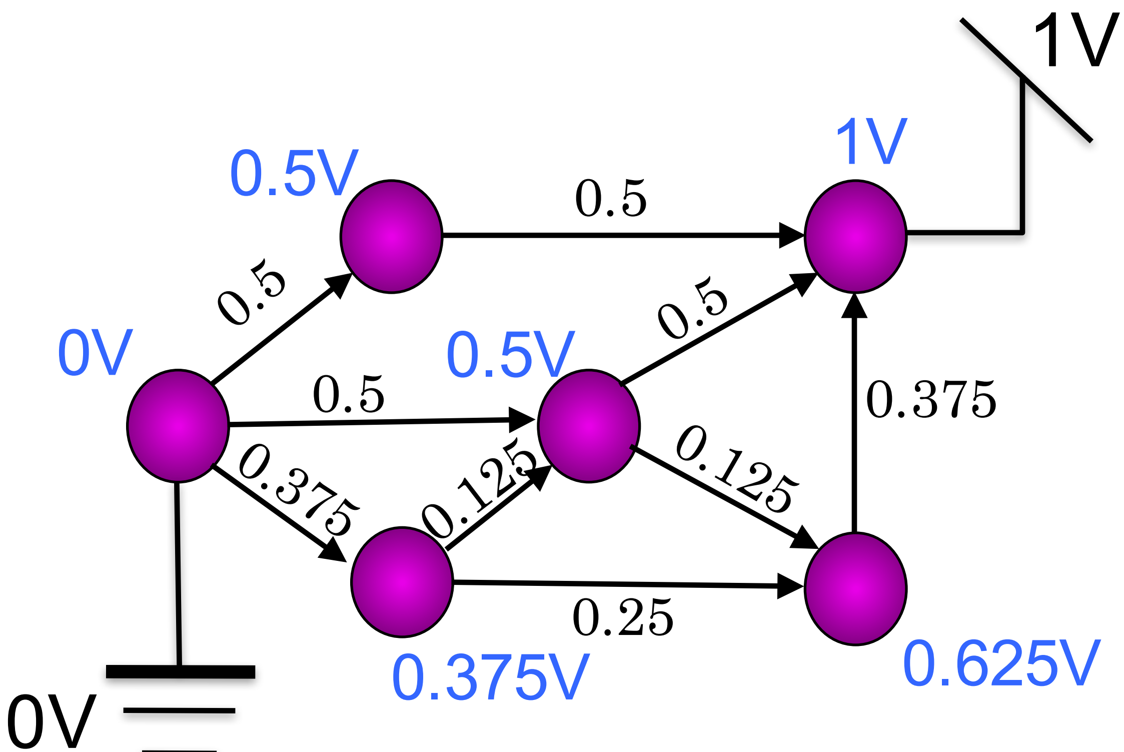


Resistor Networks

Induced voltages minimize subject to constraints.

$$\sum_{(a,b)\in E} (x(a) - x(b))^2$$

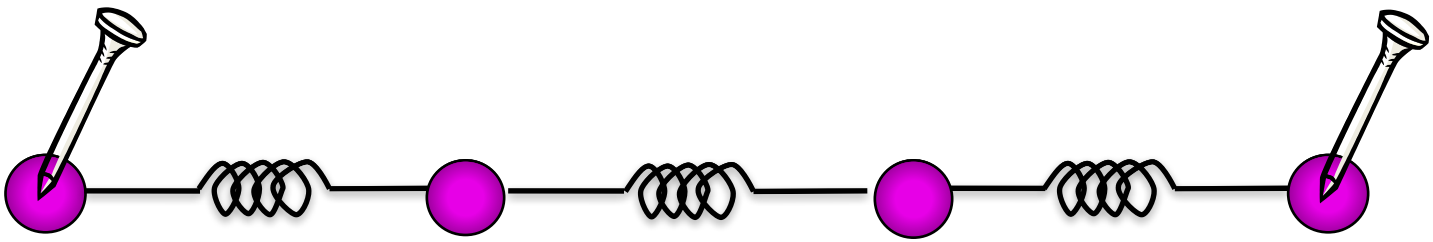
Effective resistance = 1/(current flow at one volt)



Spring Networks

View edges as rubber bands or ideal linear springs

Nail down some vertices, let rest settle

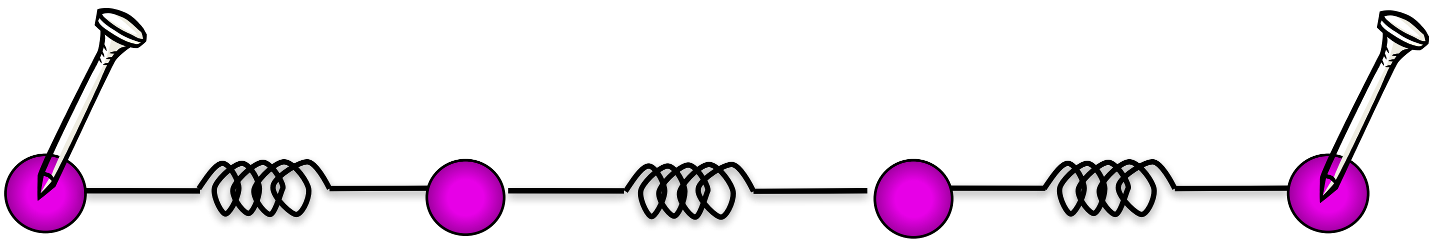


When stretched to length ℓ

potential energy is $\ell^2/2$

Spring Networks

Nail down some vertices, let rest settle

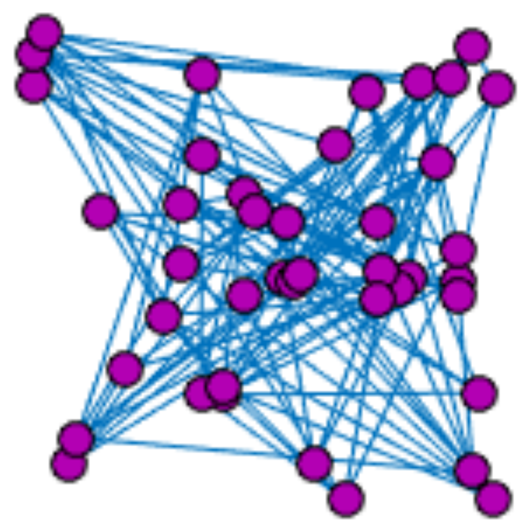


Physics: position minimizes total potential energy

$$\frac{1}{2}\sum_{(a,b)\in E}(x(a)-x(b))^2$$

subject to boundary constraints (nails)

Drawing by Spring Networks (Tutte '63)



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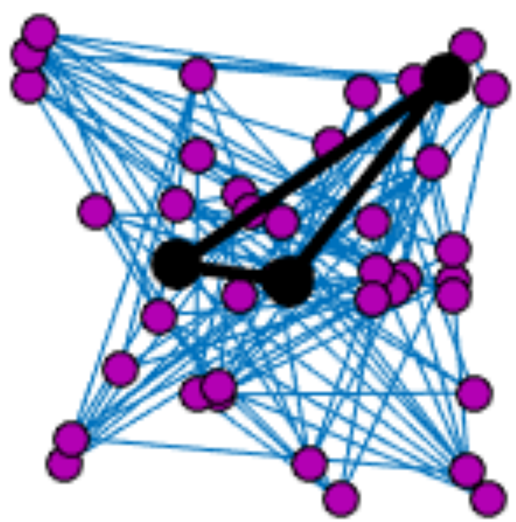

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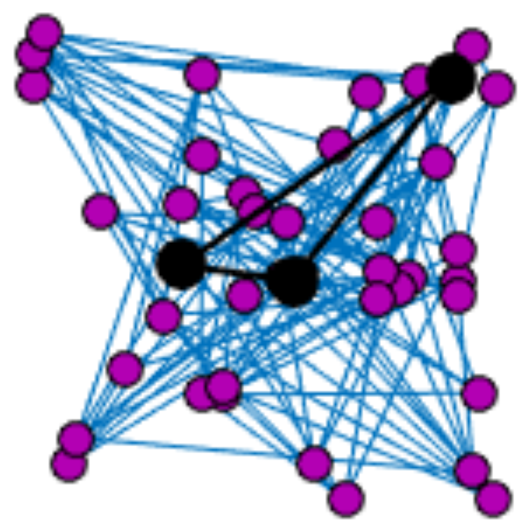

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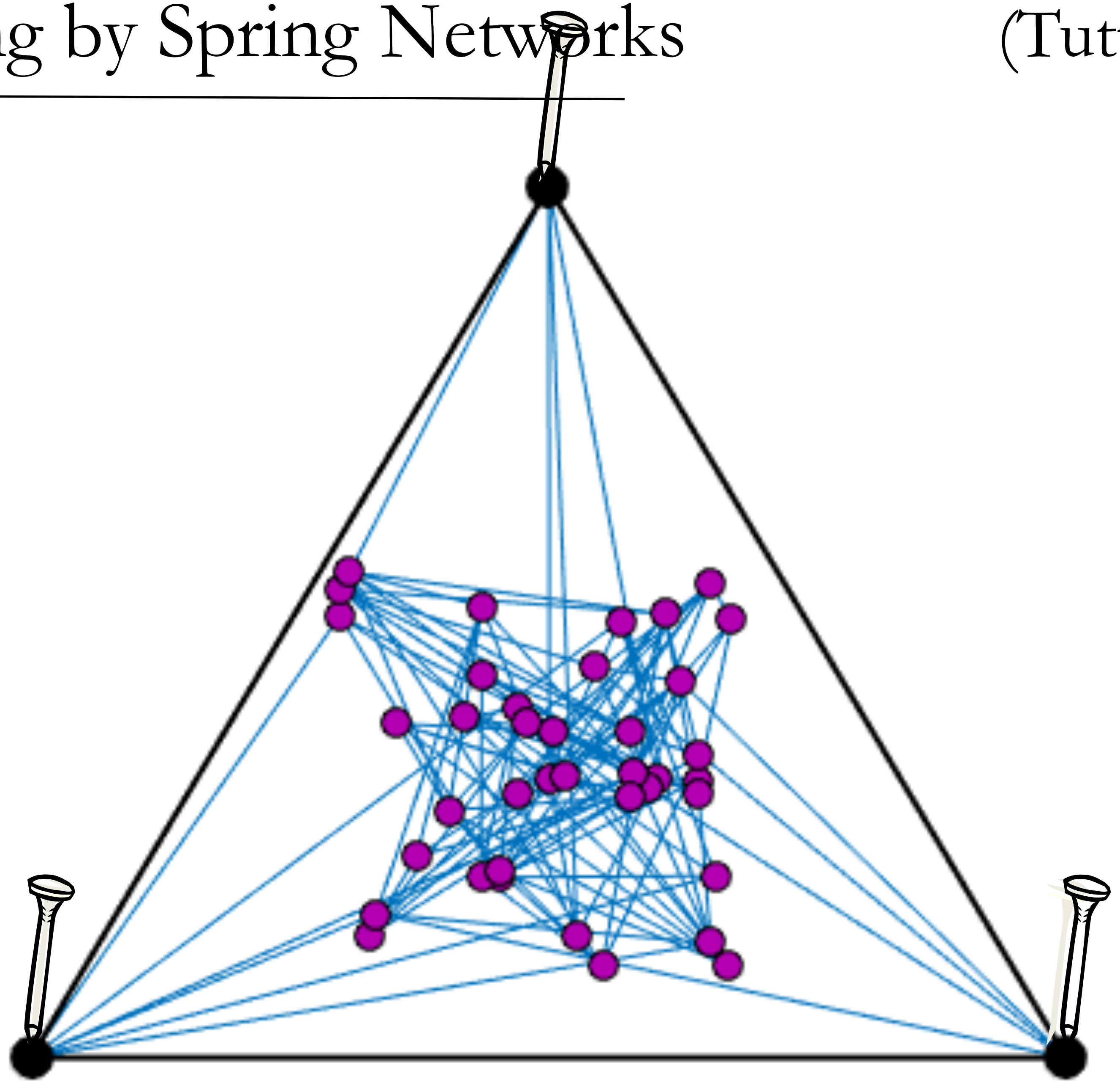

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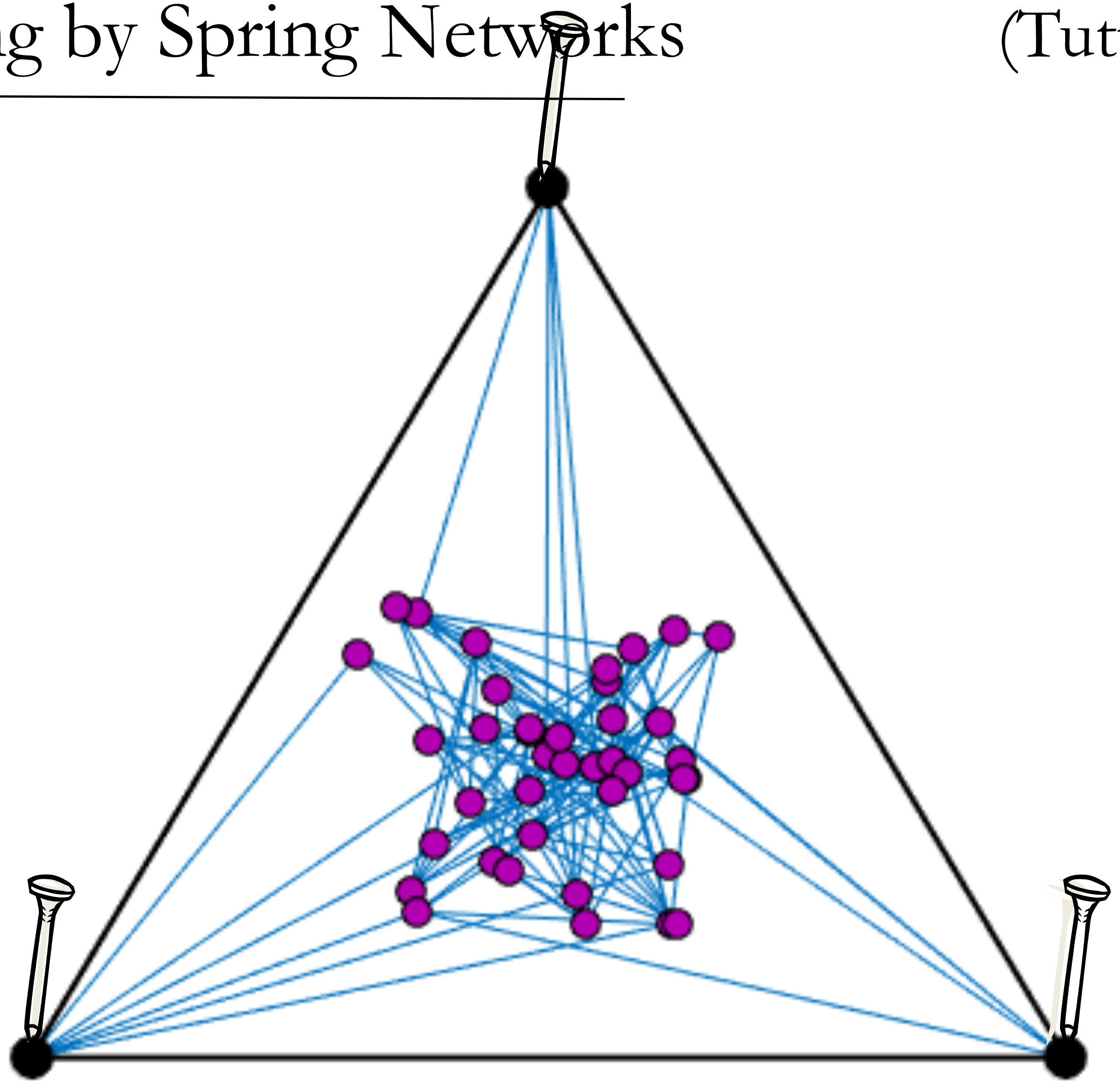
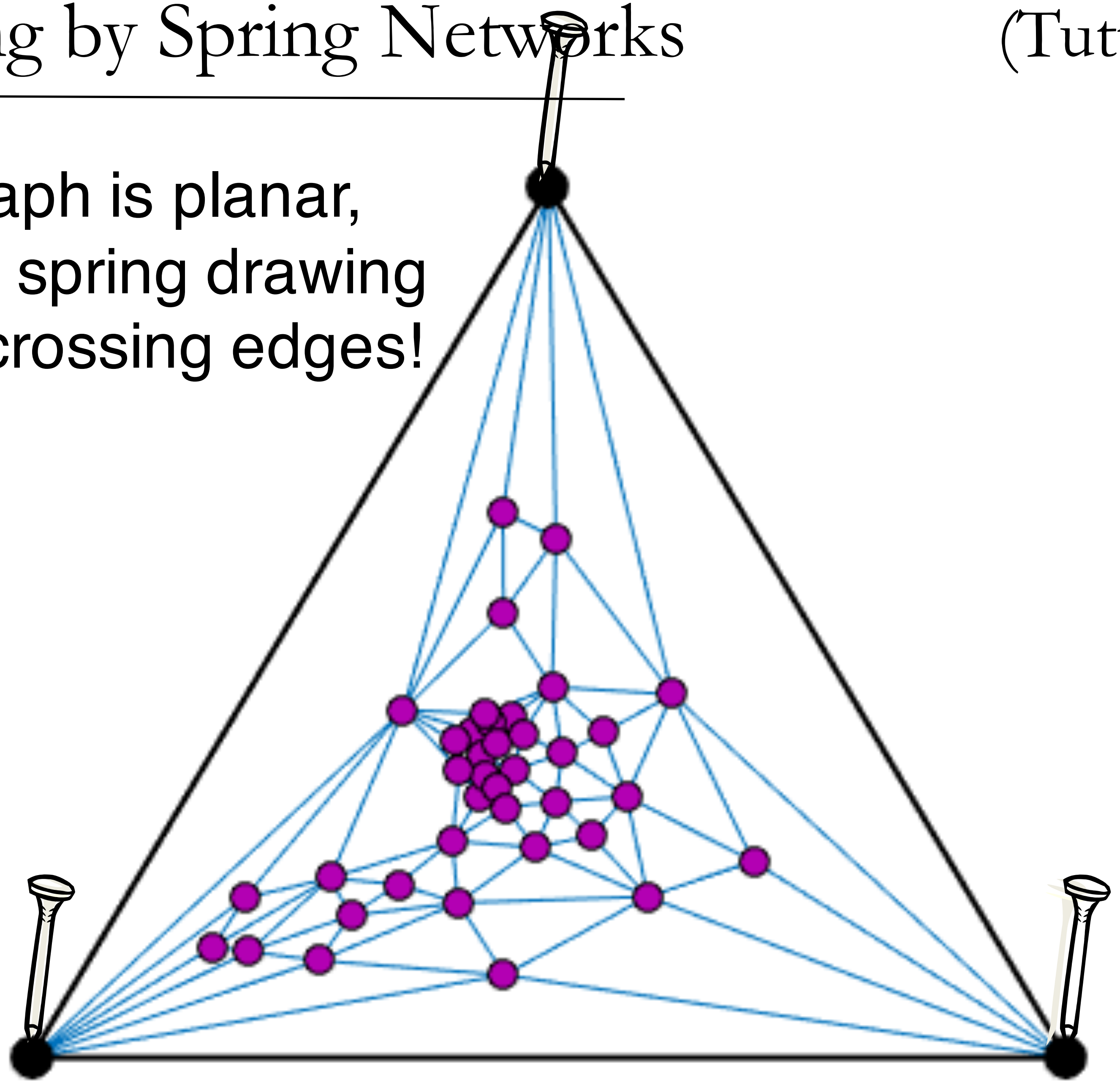


Drawing by Spring Networks (Tutte '63)

If the graph is planar, then the spring drawing has no crossing edges!



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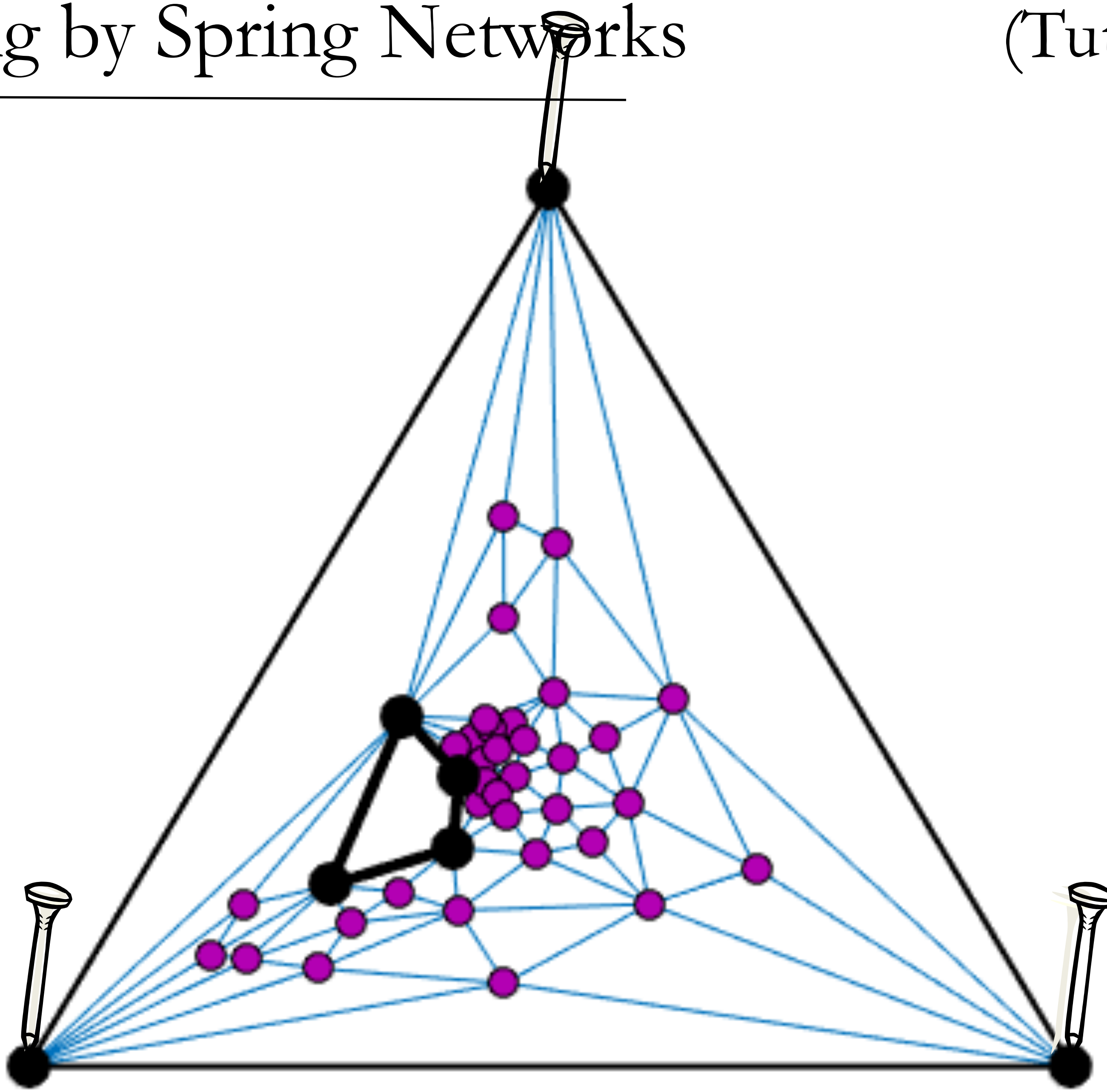

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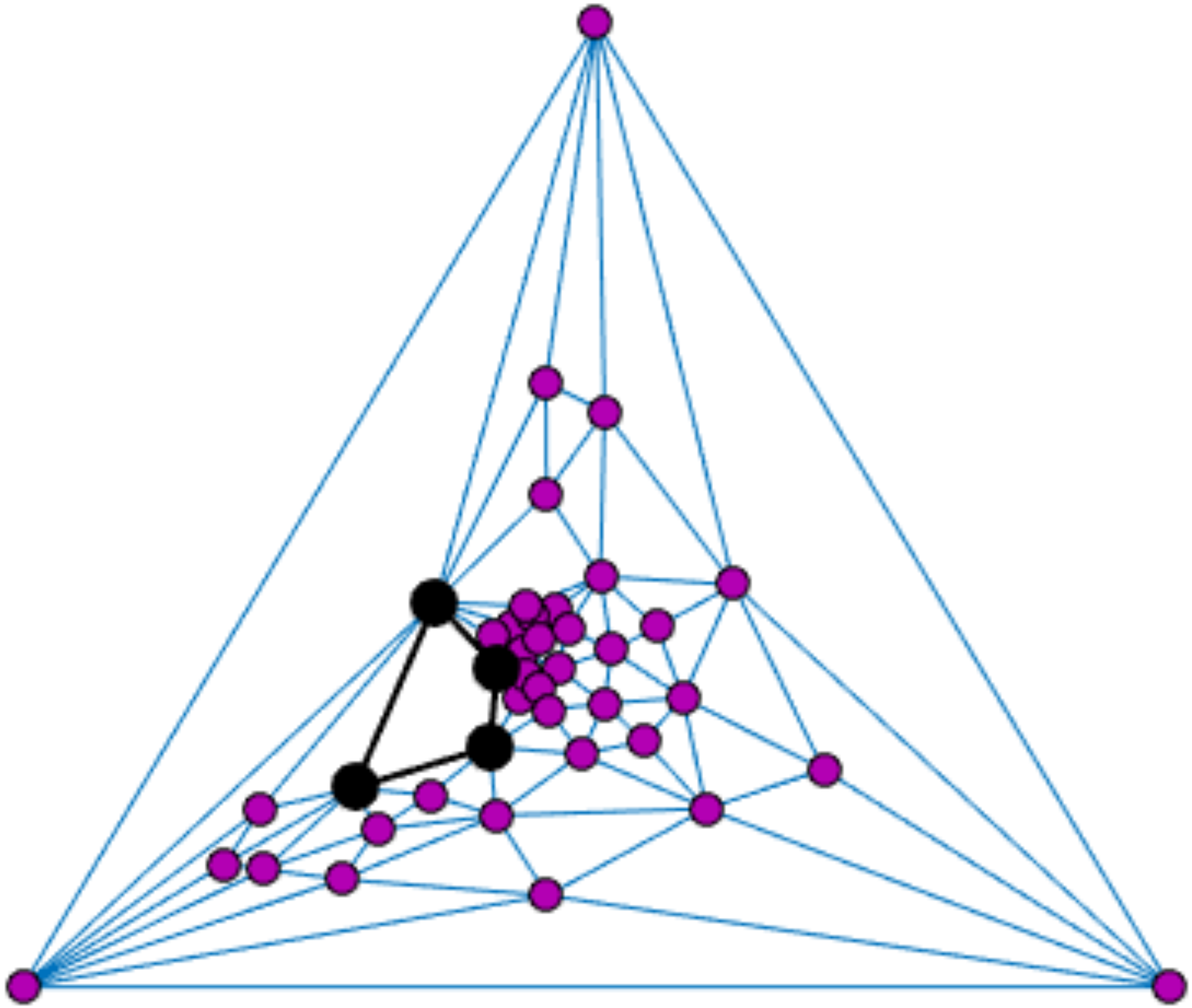

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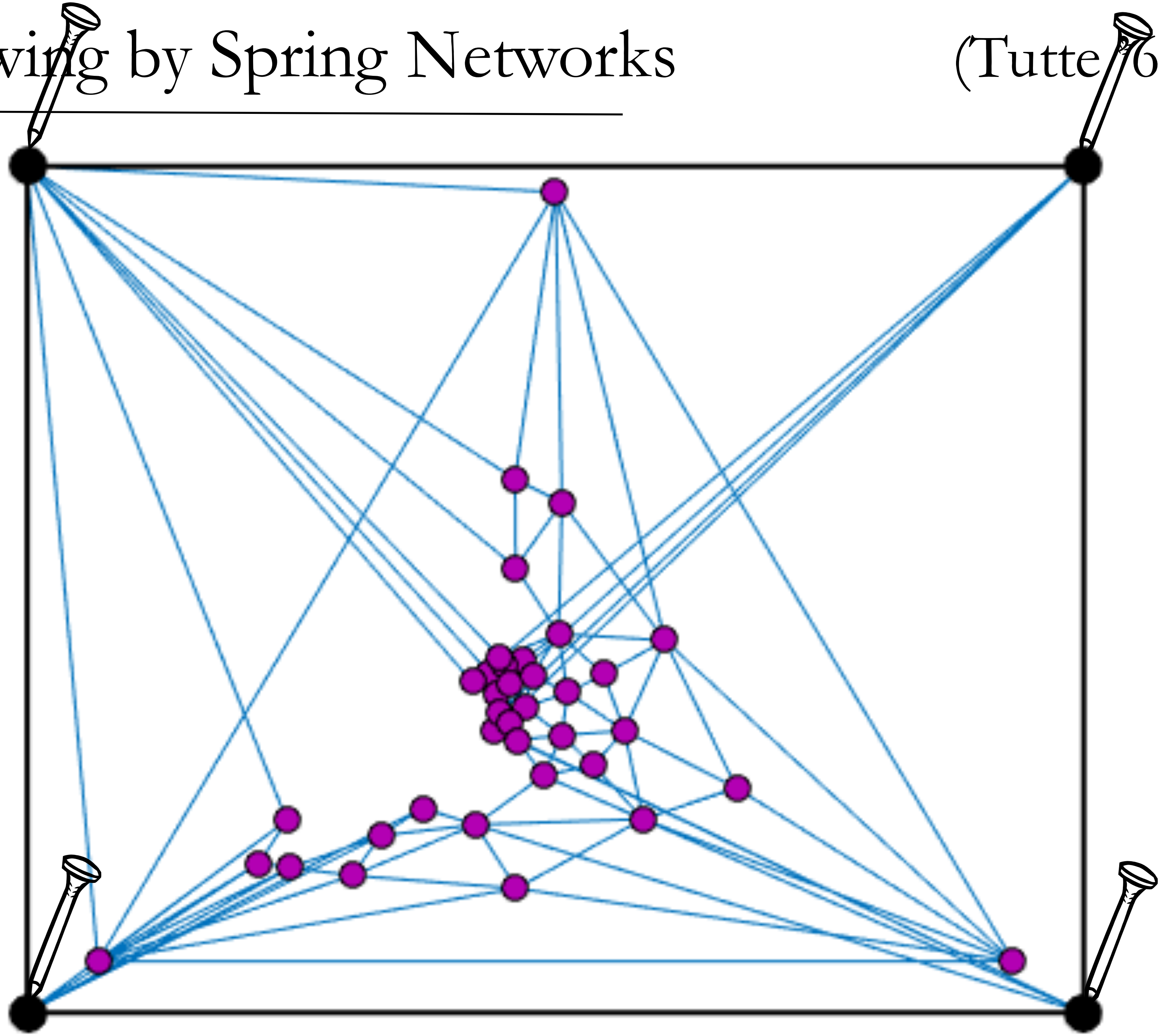

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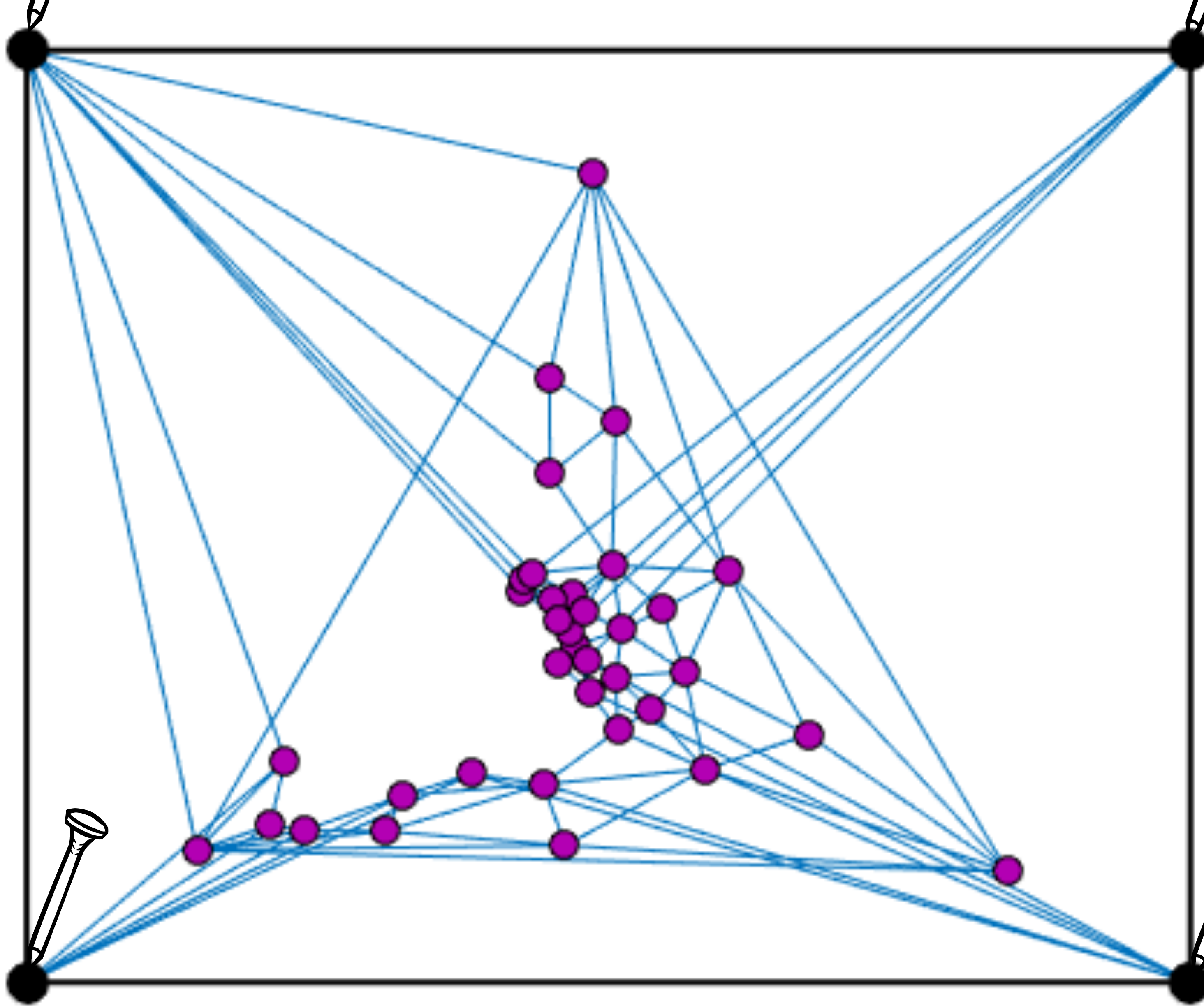

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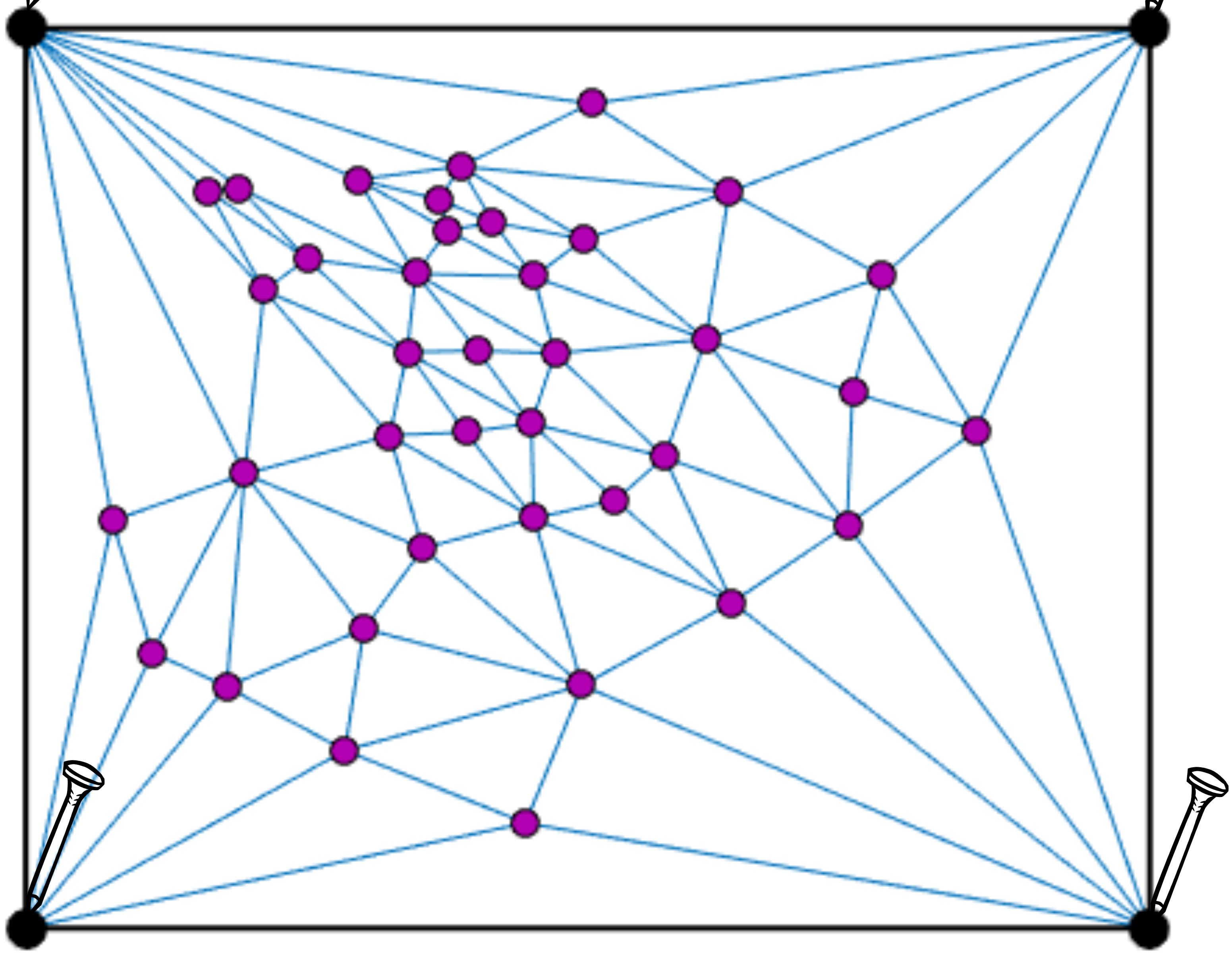

Spectral Graph Theory

A n-by-n symmetric matrix has n real eigenvalues $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ and eigenvectors $v_1, ..., v_n$ such that

$$Lv_i = \lambda_i v_i$$

These eigenvalues and eigenvectors tell us a lot about a graph!

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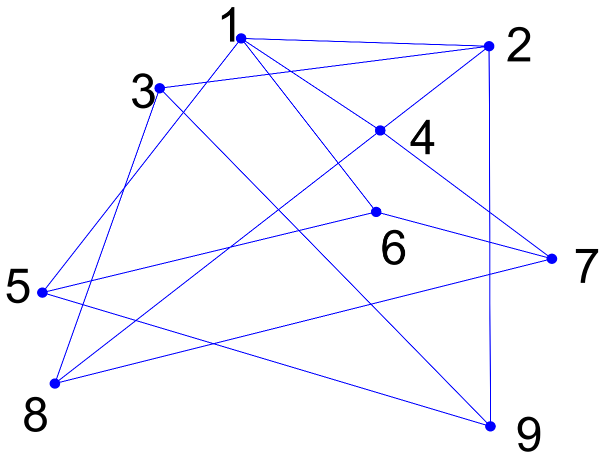
(excluding $\lambda_1 = 0, v_2 = \mathbb{1}$)

Spectral Graph Drawing

(Hall '70)

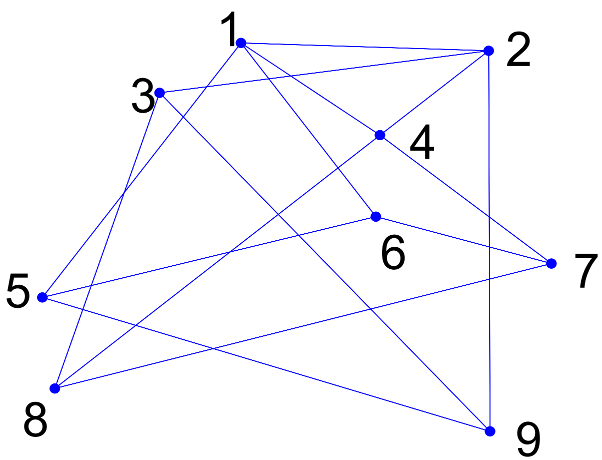


Original
Drawing

Spectral Graph Drawing

(Hall '70)

Plot vertex a at $(v_2(a), v_3(a))$
draw edges as straight lines



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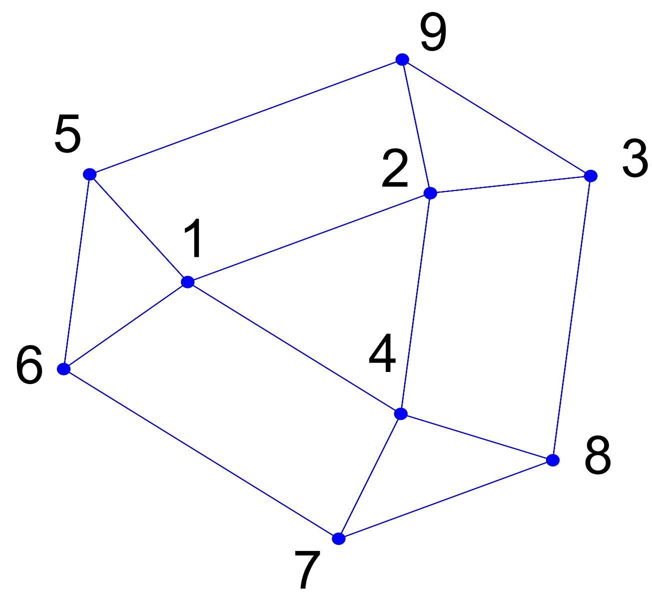

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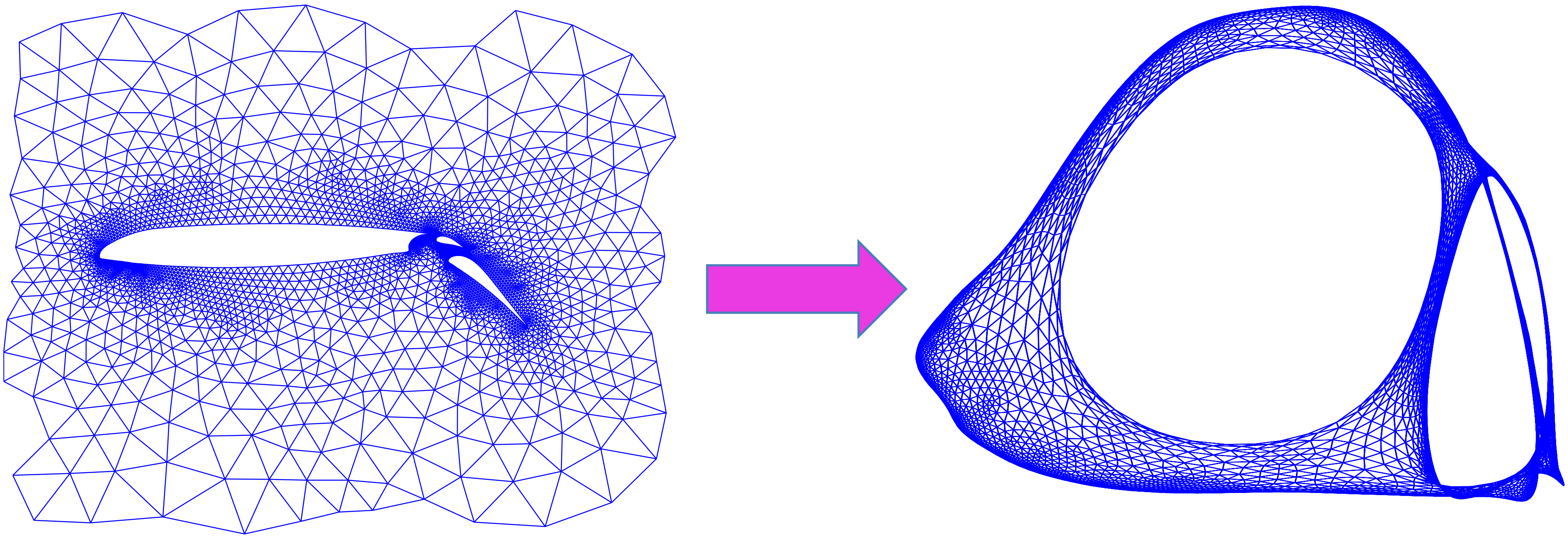

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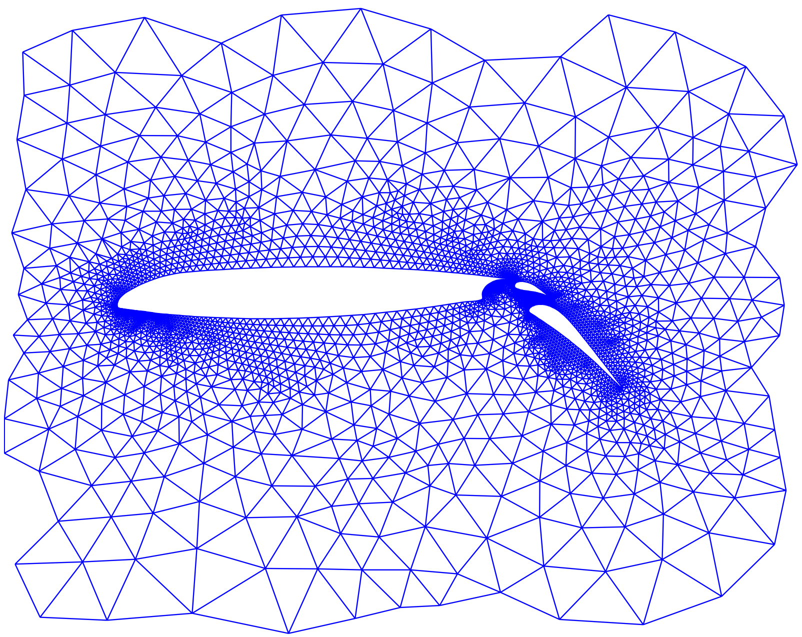
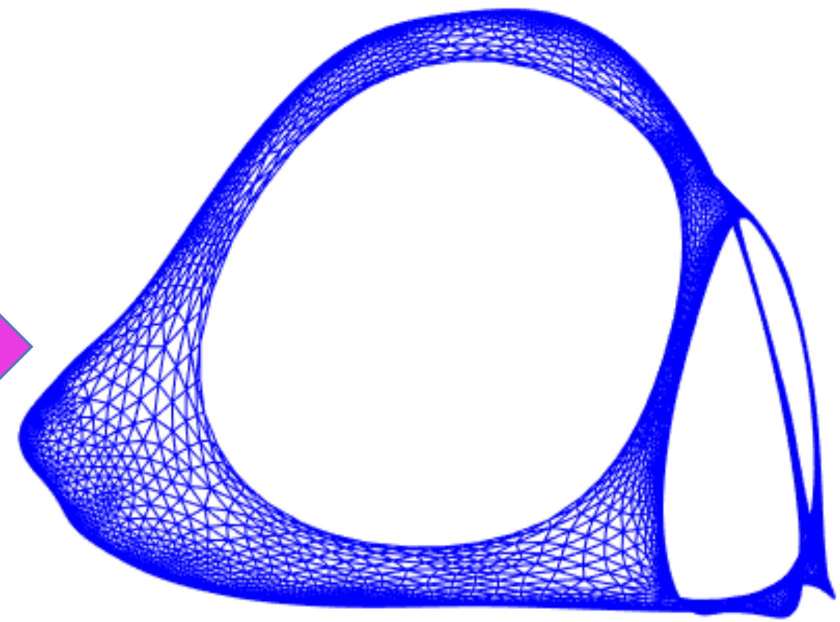
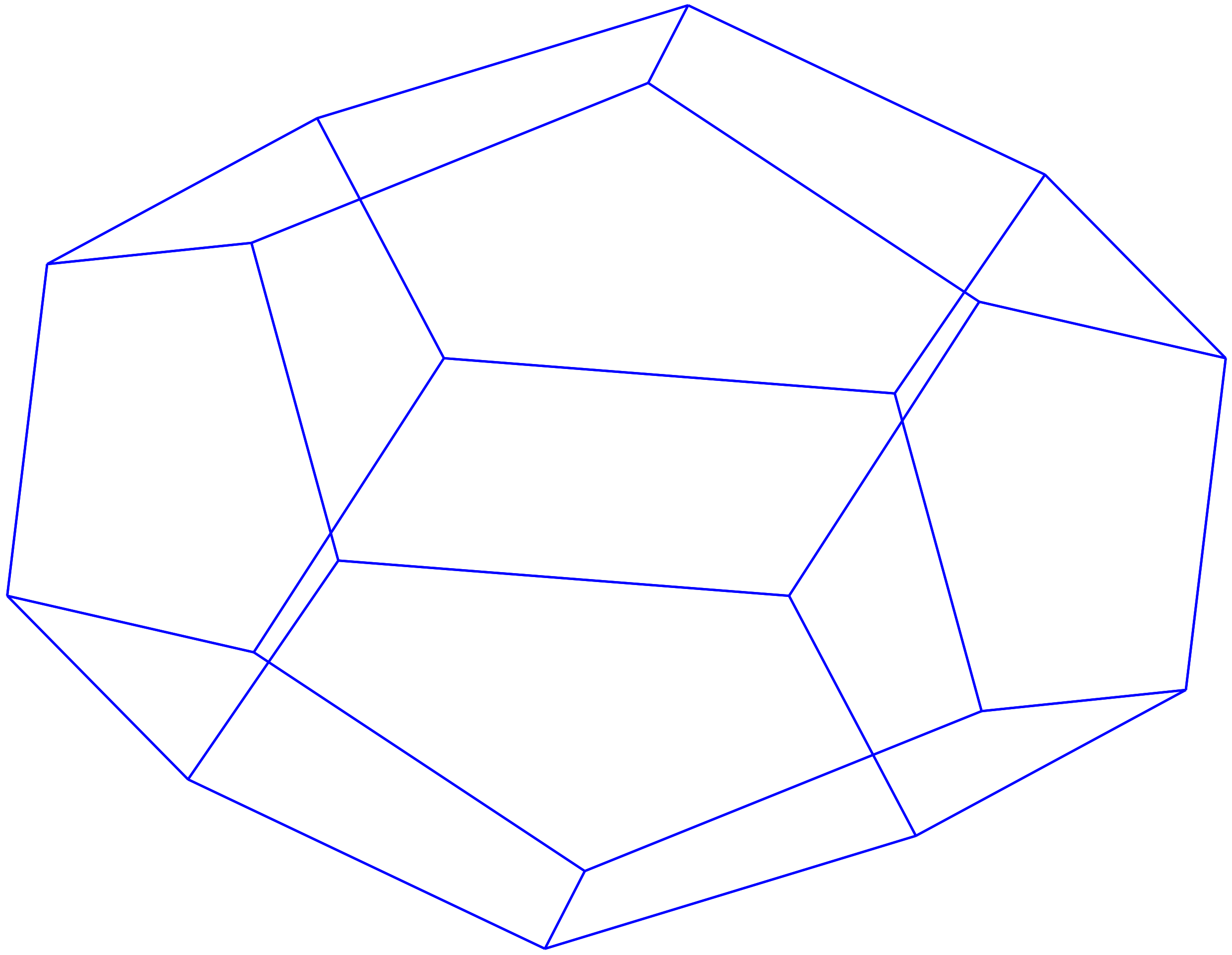




Original Drawing

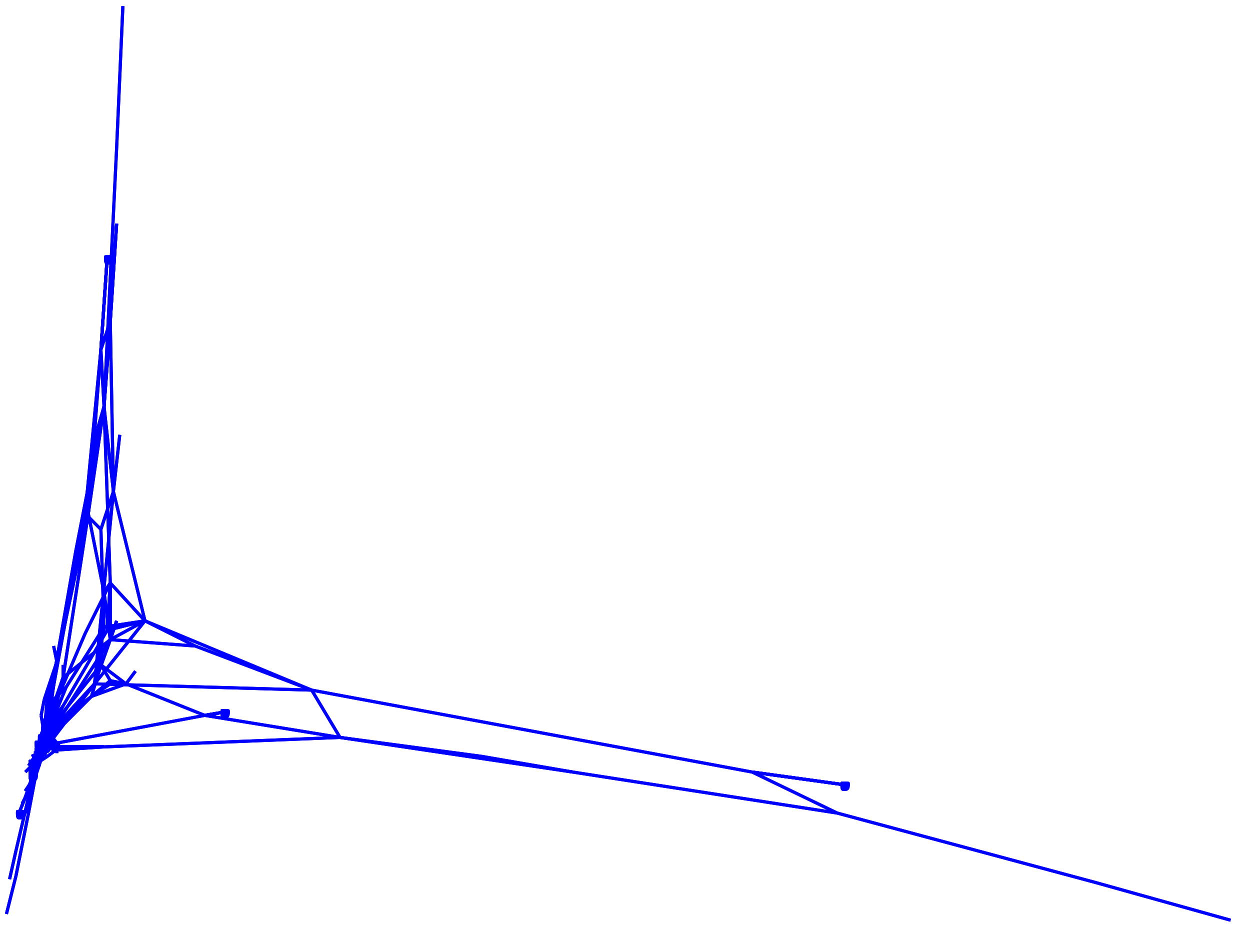
Spectral Drawing

Dodecahedron



Best embedded by first three eigenvectors

Erdos's co-authorship graph



When there is a "nice" drawing

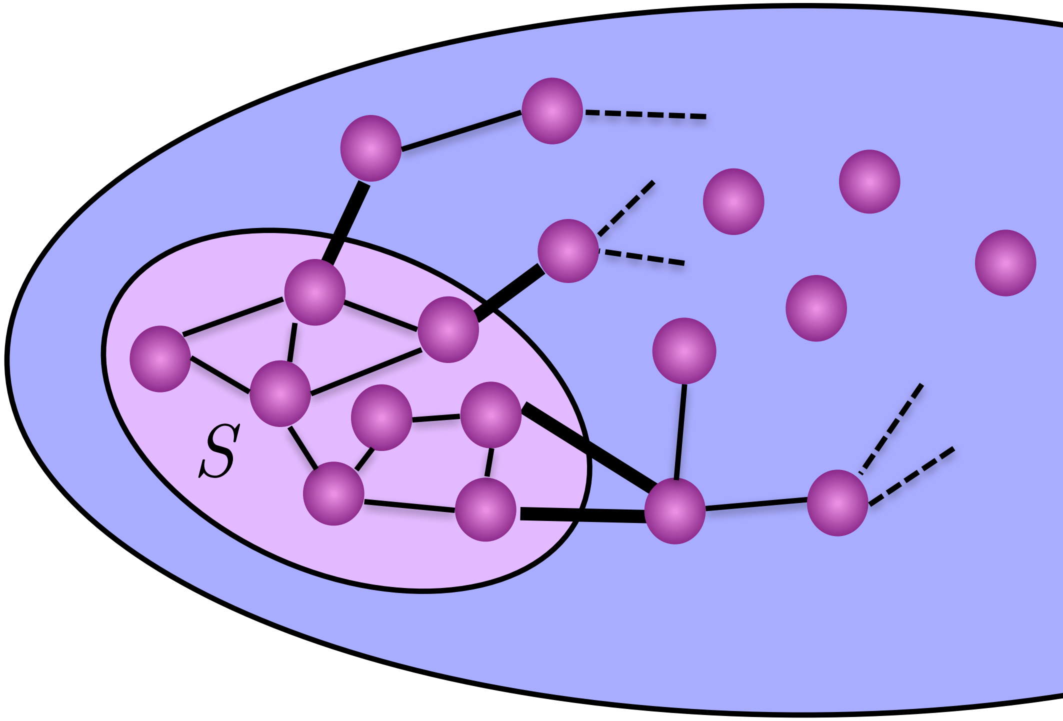
Most edges are short
Vertices are spread out and don't clump too much

⇒ λ_2 is close to 0

When λ_2 is big, say $> 10/\left|V\right|^{1/2}$
there is no nice picture of the graph

Measuring boundaries of sets

Boundary: edges leaving a set



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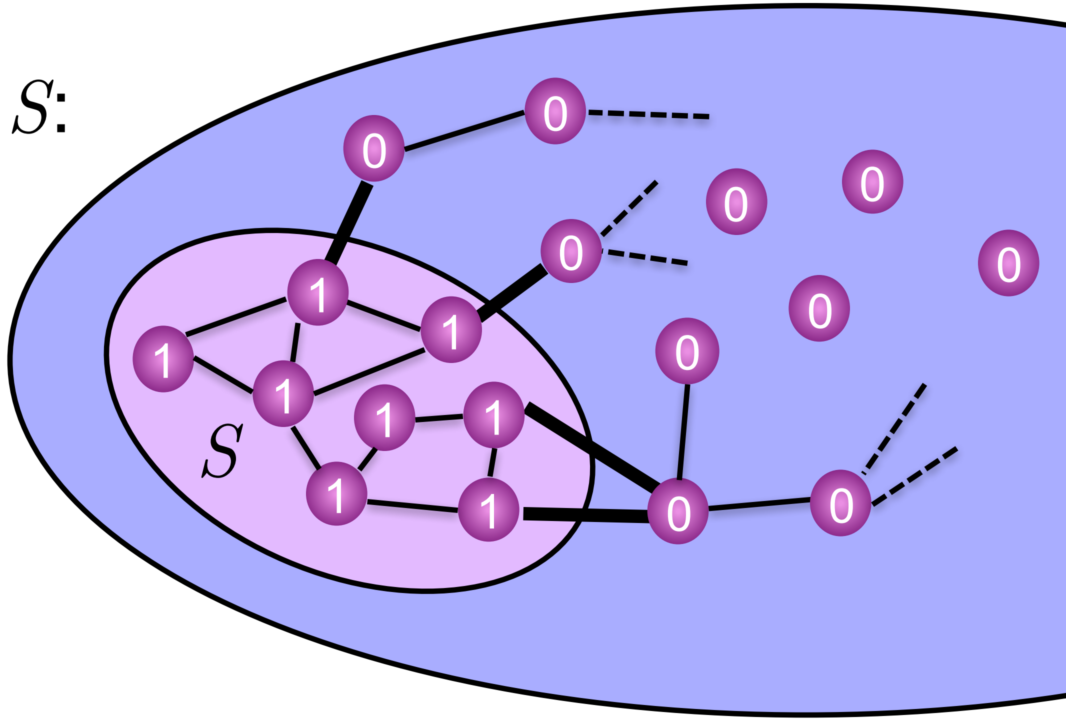
Boundary: edges leaving a set

Characteristic Vector of S:

$$x(a) = \begin{cases} 1 & a \text{ in } S \\ 0 & a \text{ not in } S \end{cases}$$



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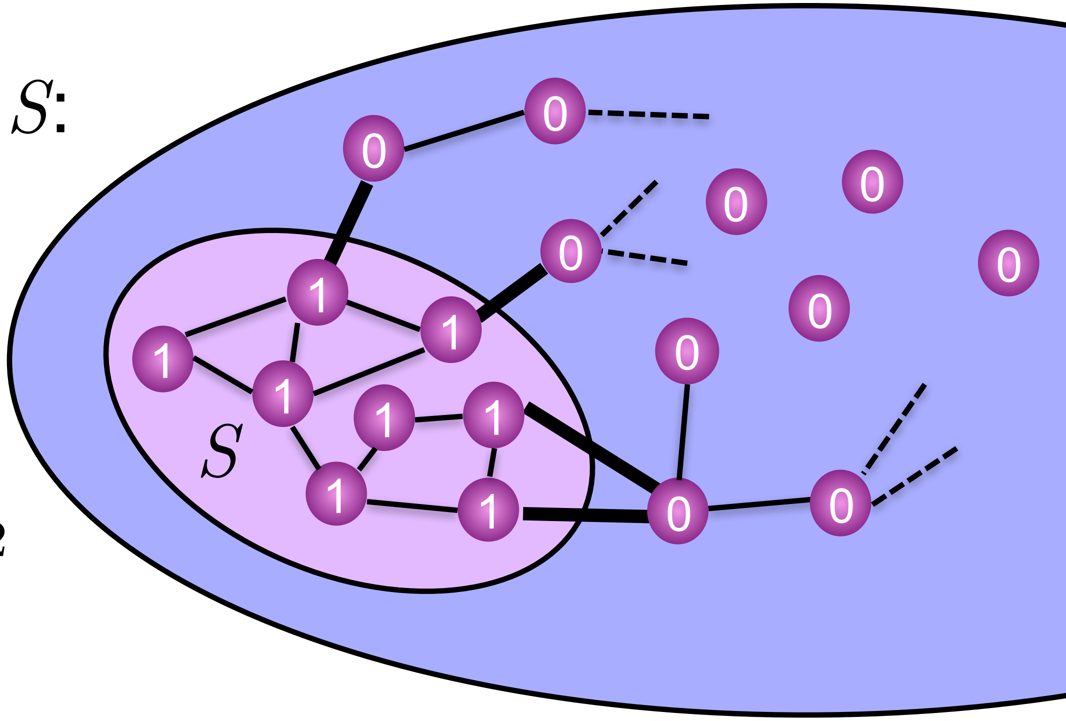
$$\sum_{(a,b)\in E} (x(a) - x(b))^2$$

$$= |\text{boundary}(S)|$$



Spectral Clustering and Partitioning

Find large sets of small boundary

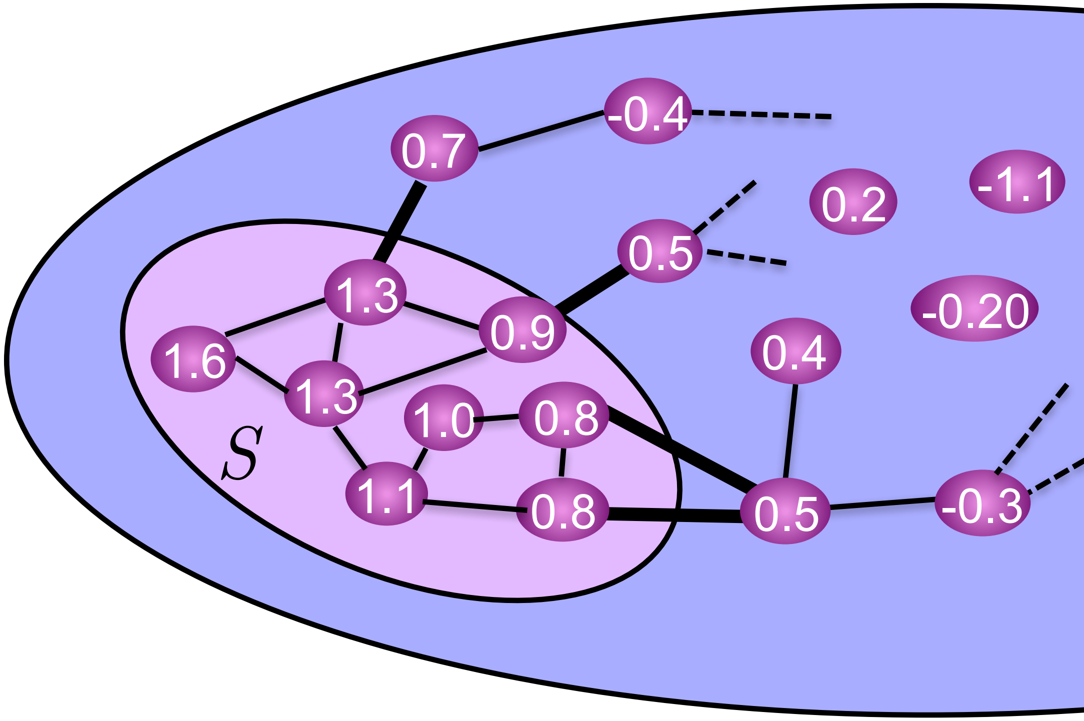
Heuristic to find

x with $x^T L_G x$ small

Compute eigenvector

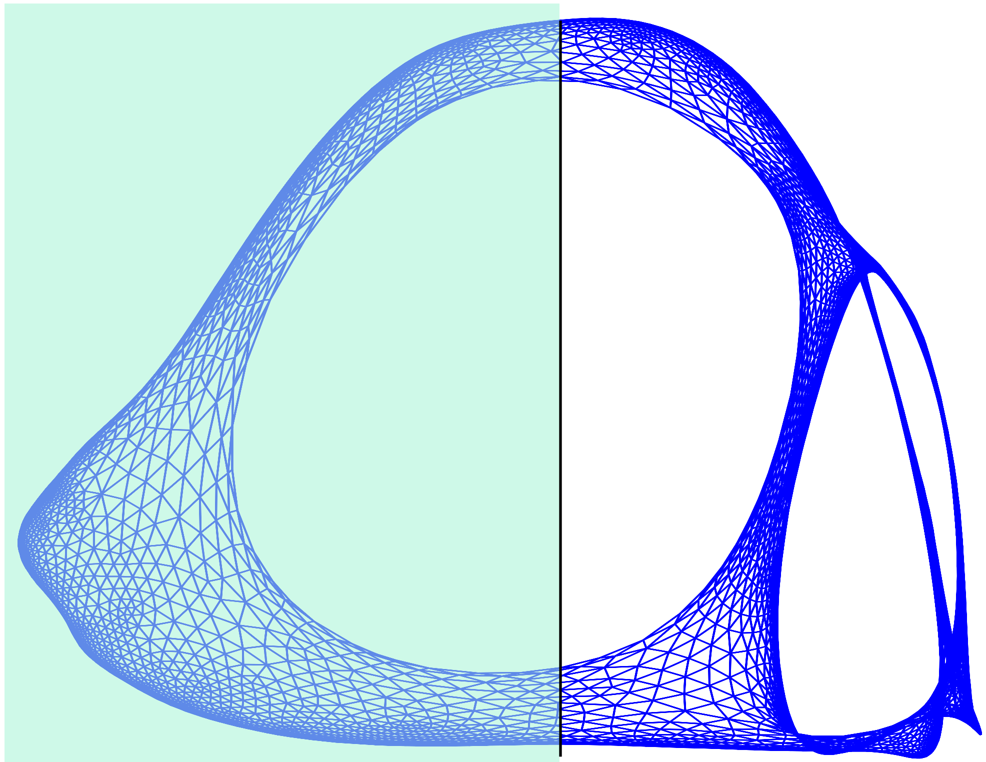
$$L_G v_2 = \lambda_2 v_2$$

Consider the level sets



Spectral Partitioning

(Donath-Hoffman ‘72, Barnes ‘82, Hagen-Kahng ’92)



$S = \{a : v_2(a) \leq t\}$ for some t

Cheeger’s inequality implies good approximation

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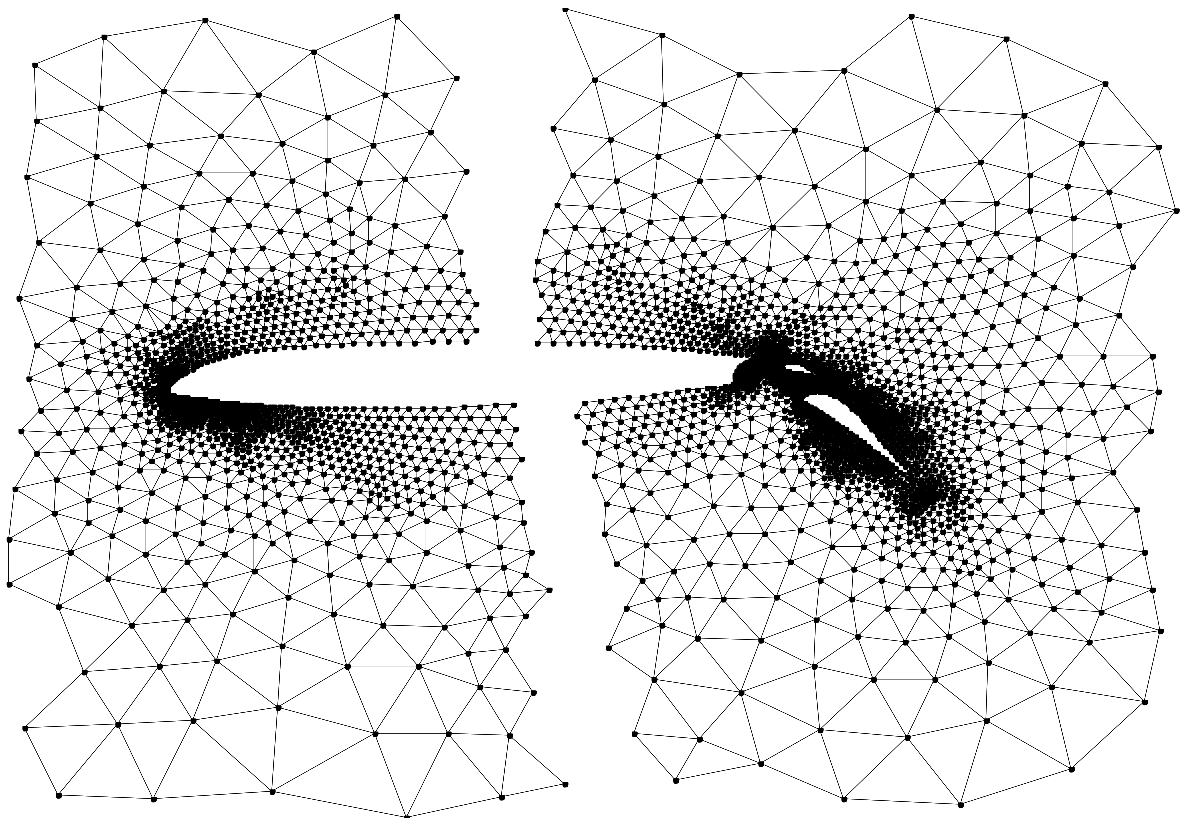

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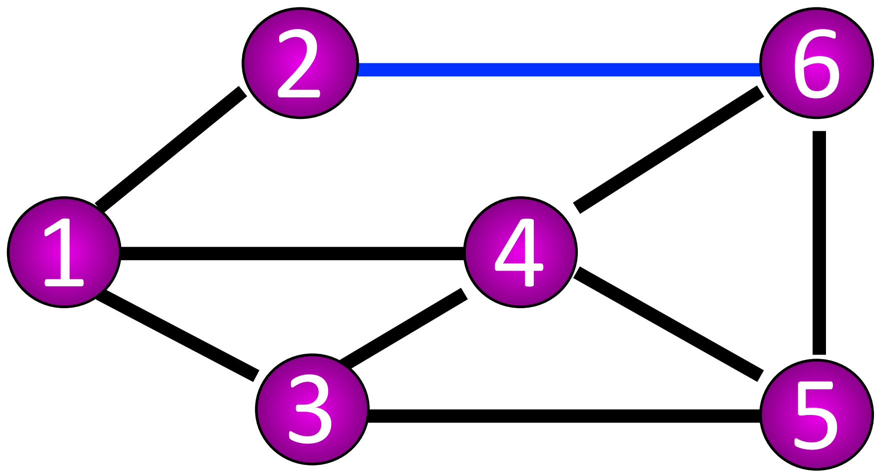
The Laplacian Matrix of a Graph



$$\begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix}$$

Symmetric

Non-positive off-diagonals

Diagonally dominant

The Laplacian Matrix of a Graph

$$x^T L_G x = \sum_{(a,b) \in E} w_{a,b} (x(a) - x(b))^2$$

$$L_G = \sum_{(a,b) \in E} w_{a,b} L_{a,b}$$

$$\begin{aligned} L_{1,2} &= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \end{aligned}$$

Quickly Solving Laplacian Equations

S,Teng '04: Using low-stretch trees and sparsifiers

$$O(m \log^c n \log \epsilon^{-1})$$

Where m is number of non-zeros and n is dimension

Quickly Solving Laplacian Equations

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Koutis, Miller, Peng '11: Low-stretch trees and sampling

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Cohen, Kyng, Pachocki, Peng, Rao '14:

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Good code:

- LAMG (lean algebraic multigrid) – Livne-Brandt
- CMG (combinatorial multigrid) – Koutis

Quickly Solving Laplacian Equations

S,Teng '04: Using low-stretch trees and sparsifiers

$$O(m \log^c n \log \epsilon^{-1})$$

An ϵ-accurate solution to $L_G x = b$
is an $\widetilde{x}$ satisfying

$$\|\widetilde{x} - x\|_{L_G} \leq \epsilon \|x\|_{L_G}$$

where $\|v\|_{L_G} = \sqrt{v^T L_G v} = ||L_G^{1/2} v||$

Quickly Solving Laplacian Equations

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Allows fast computation of eigenvectors
corresponding to small eigenvalues.

Laplacians in Linear Programming

Laplacians appear when solving Linear Programs on on graphs by Interior Point Methods

Maximum and Min-Cost Flow (Daitch, S ’08, Mądry ‘13)

Shortest Paths (Cohen, Mądry, Sankowski, Vladu ‘16)

Isotonic Regression (Kyng, Rao, Sachdeva ‘15)

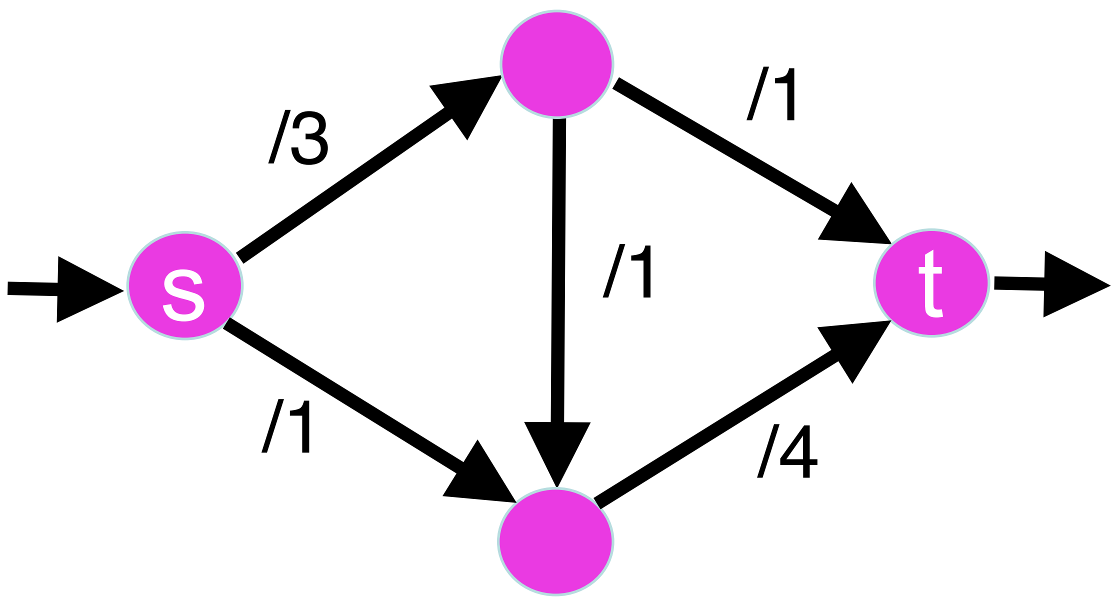
Lipschitz Learning : regularized interpolation on graphs (Kyng, Rao, Sachdeva, S ‘15)

Interior Point Method for Maximum s-t Flow

maximize $f^{out}(s)$

subject to $\quad f^{out}(a) = f^{in}(a), \quad \forall a \notin \{s, t\}$

$$0 \leq f(a,b) \leq c(a,b), \quad \forall (a,b) \in E$$



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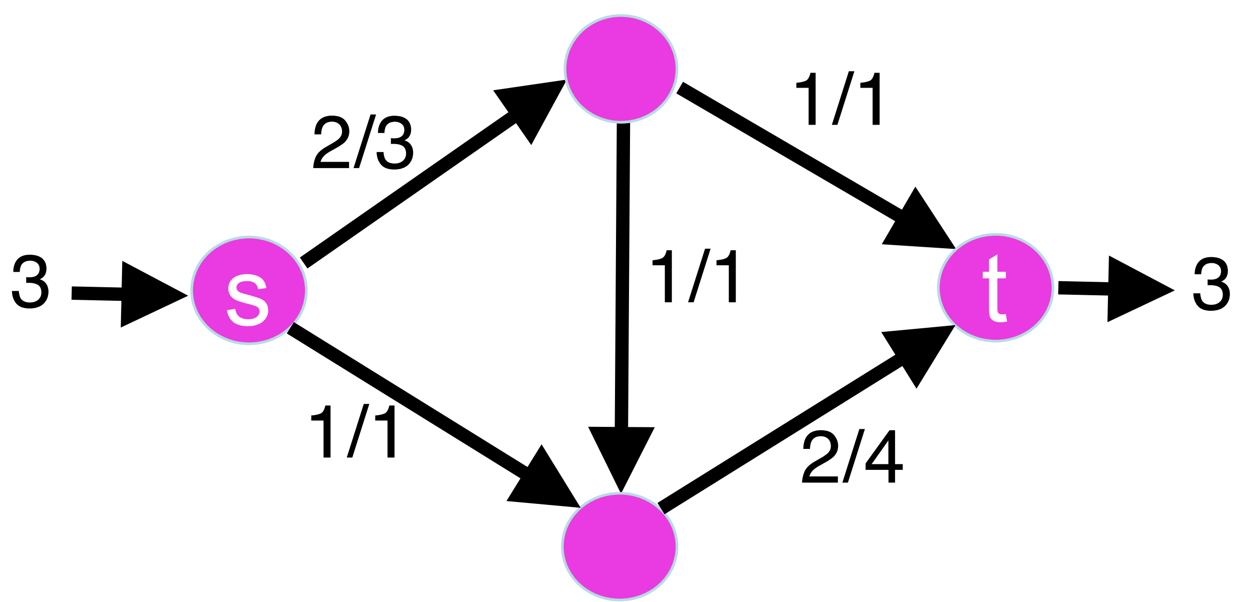

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Multiple calls with varying weights $w_{a,b}$

maximize $f^{out}(s)$

subject to $\quad f^{out}(a) = f^{in}(a), \quad \forall a \notin \{s, t\}$

$$\sum_{(a,b) \in E} w_{a,b} f(a,b)^2 \le C$$

Spectral Sparsification

Every graph can be approximated
by a sparse graph with a similar Laplacian

Approximating Graphs

A graph H is an ϵ-approximation of G if

for all x
$$\frac{1}{1+\epsilon} \leq \frac{x^T L_H x}{x^T L_G x} \leq 1+\epsilon$$

$$L_H \approx_\epsilon L_G$$

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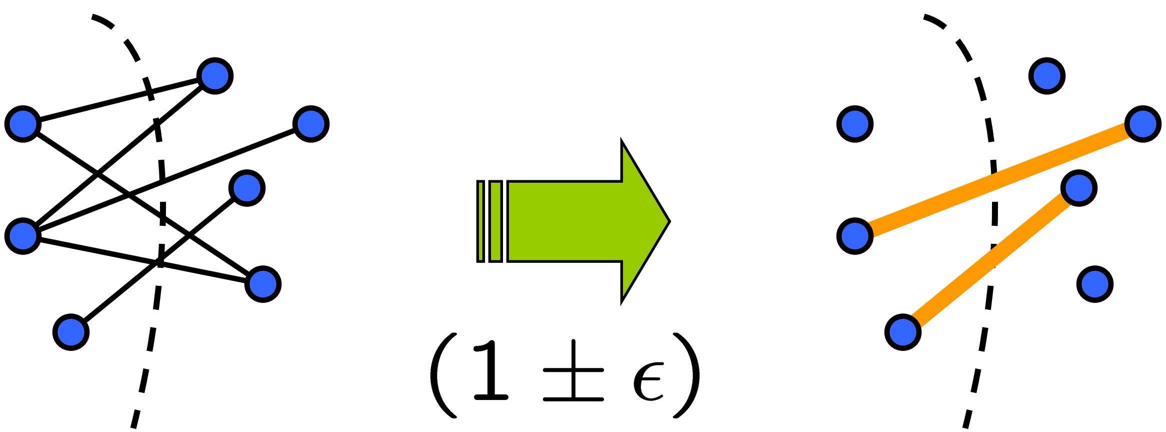
Preserves boundaries of every set



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Solutions to linear equations are similar

$$L_H \approx_\epsilon L_G \iff L_H^{-1} \approx_\epsilon L_G^{-1}$$

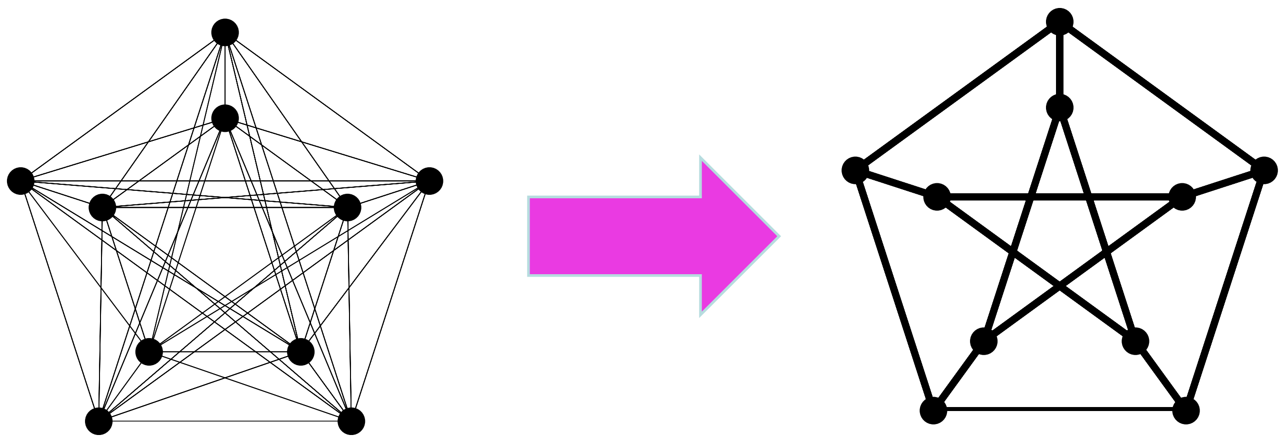
As are effective resistances

Expanders Sparsify Complete Graphs

Yield good LDPC codes

Every set of vertices has large boundary

λ_2 is large



Random regular graphs are usually expanders

Sparsification by Random Sampling

Assign a probability $p_{a,b}$ to each edge (a,b)

Include edge (a,b) in H with probability $p_{a,b}$.

If include edge (a,b), give it weight $w_{a,b}/p_{a,b}$

$$\mathbb{E}\left[\ L_H\ \right] = \sum_{(a,b)\in E} p_{a,b}(w_{a,b}/p_{a,b})L_{a,b} = L_G$$

Sparsification by Random Sampling

Choose $p_{a,b}$ to be $w_{a,b}$ times the
effective resistance between a and b.

Low resistance between a and b means there are many alternate routes for current to flow and that the edge is not critical.

Proof by random matrix concentration bounds
(Rudelson, Ahlswede-Winter, Tropp, etc.)

Only need $O(n \log n / \epsilon^2)$ edges

(S, Srivastava '08)

Optimal Graph Sparsification?

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

$$L_G \approx_\epsilon L_H \quad \text{and} \quad |F| \leq (2+\epsilon)^2 n/\epsilon^2$$

Is within a factor of 2 of how well
Ramanujan expanders approximate complete graphs

(Batson, S, Srivastava '09)

Approximate Gaussian Elimination

(Kyng & Sachdeva '16)

Gaussian Elimination:

compute upper triangular U so that

$$L_G = U^T U$$

Approximate Gaussian Elimination:

compute sparse upper triangular U so that

$$L_G \approx U^T U$$

Additive view of Gaussian Elimination

Find U, upper triangular matrix, s.t $U^\top U = A$

$$A = \begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 14 & 0 \\ -4 & -1 & 0 & 7 \end{pmatrix}$$

Additive view of Gaussian Elimination

$$\begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 14 & 0 \\ -4 & -1 & 0 & 7 \end{pmatrix}$$

Find the rank-1 matrix that agrees on the first row and column.

$$\begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 1 & 2 & 1 \\ -8 & 2 & 4 & 2 \\ -4 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix}^{\top}$$

Additive view of Gaussian Elimination

$$\begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 14 & 0 \\ -4 & -1 & 0 & 7 \end{pmatrix} - \begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 1 & 2 & 1 \\ -8 & 2 & 4 & 2 \\ -4 & 1 & 2 & 1 \end{pmatrix}$$

Subtract the rank 1 matrix.

We have **eliminated the first variable.**

Additive view of Gaussian Elimination

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 10 & -2 \\ 0 & -2 & -2 & 6 \end{pmatrix}$$

Additive view of Gaussian Elimination

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 10 & -2 \\ 0 & -2 & -2 & 6 \end{pmatrix}$$

Find the rank-1 matrix that agrees on the **next** row and column.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 1 & 1 \\ 0 & -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}^{\top}$$

Additive view of Gaussian Elimination

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 10 & -2 \\ 0 & -2 & -2 & 6 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 1 & 1 \\ 0 & -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & -3 \\ 0 & 0 & -3 & 5 \end{pmatrix}$$

Subtract the rank 1 matrix.

We have **eliminated the second variable.**

Additive view of Gaussian Elimination

$$A = \begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 14 & 0 \\ -4 & -1 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix}^{\top} + \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}^{\top} + \begin{pmatrix} 0 \\ 0 \\ 3 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 3 \\ -1 \end{pmatrix}^{\top} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}^{\top}$$

Running time proportional to sum of squares of number of non-zeros in these vectors.

Additive view of Gaussian Elimination

$$A = \begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 14 & 0 \\ -4 & -1 & 0 & 7 \end{pmatrix}$$

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$$= \begin{pmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -2 & -1 & 3 & 0 \\ -1 & -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 4 & -1 & -2 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

Additive view of Gaussian Elimination

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$$= \begin{pmatrix} 4 & -1 & -2 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}^\top \begin{pmatrix} 4 & -1 & -2 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix} = U^\top U$$

Gaussian Elimination of Laplacians

If this is a Laplacian, then so is this

$$\begin{pmatrix} 16 & -4 & -8 & -4 \\ -4 & 5 & 0 & -1 \\ -8 & 0 & 14 & 0 \\ -4 & -1 & 0 & 7 \end{pmatrix} - \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ -2 \\ -1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & -2 & -2 \\ 0 & -2 & 10 & -2 \\ 0 & -2 & -2 & 6 \end{pmatrix}$$

Gaussian Elimination of Laplacians

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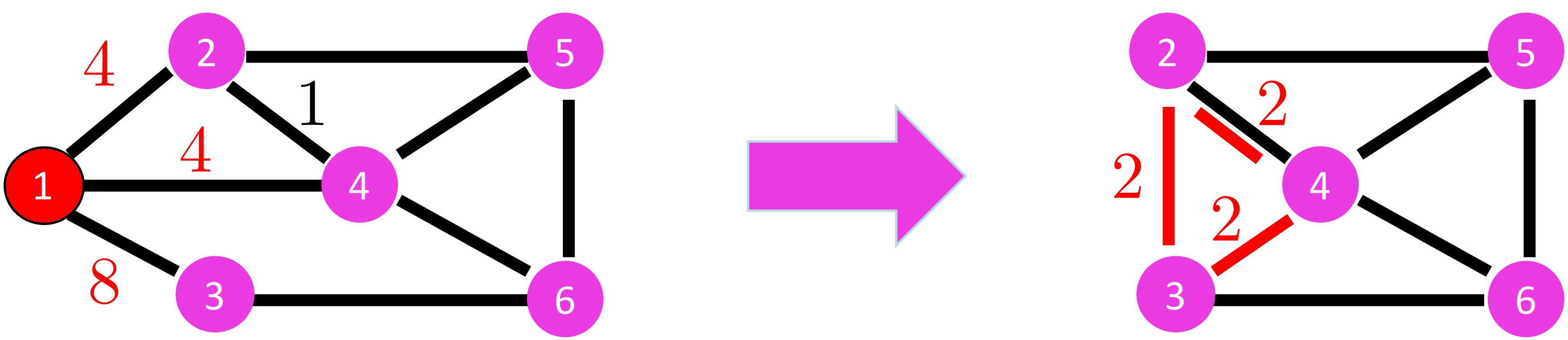
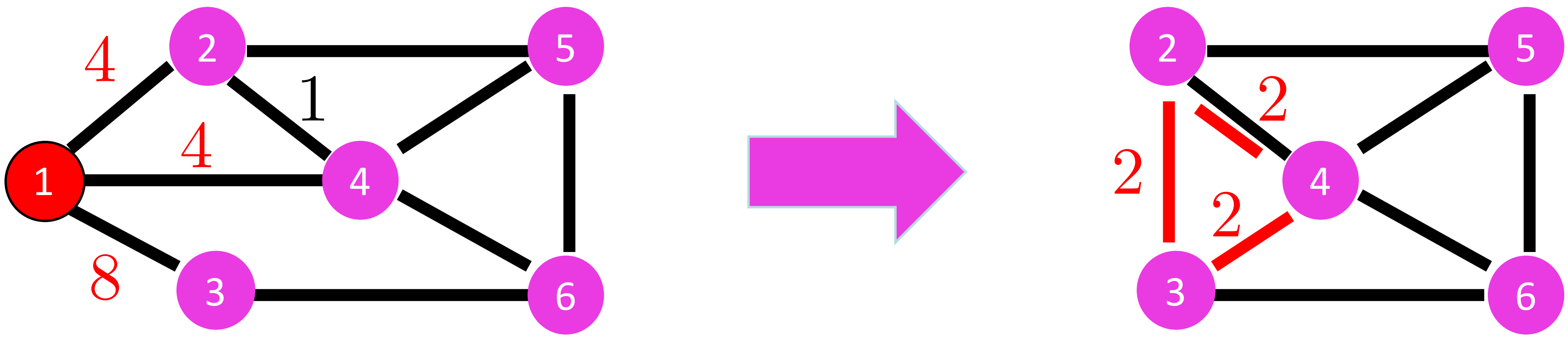
When eliminate a node, add a clique on its neighbors



Approximate Gaussian Elimination

(Kyng & Sachdeva '16)

1. when eliminate a node, add a clique on its neighbors



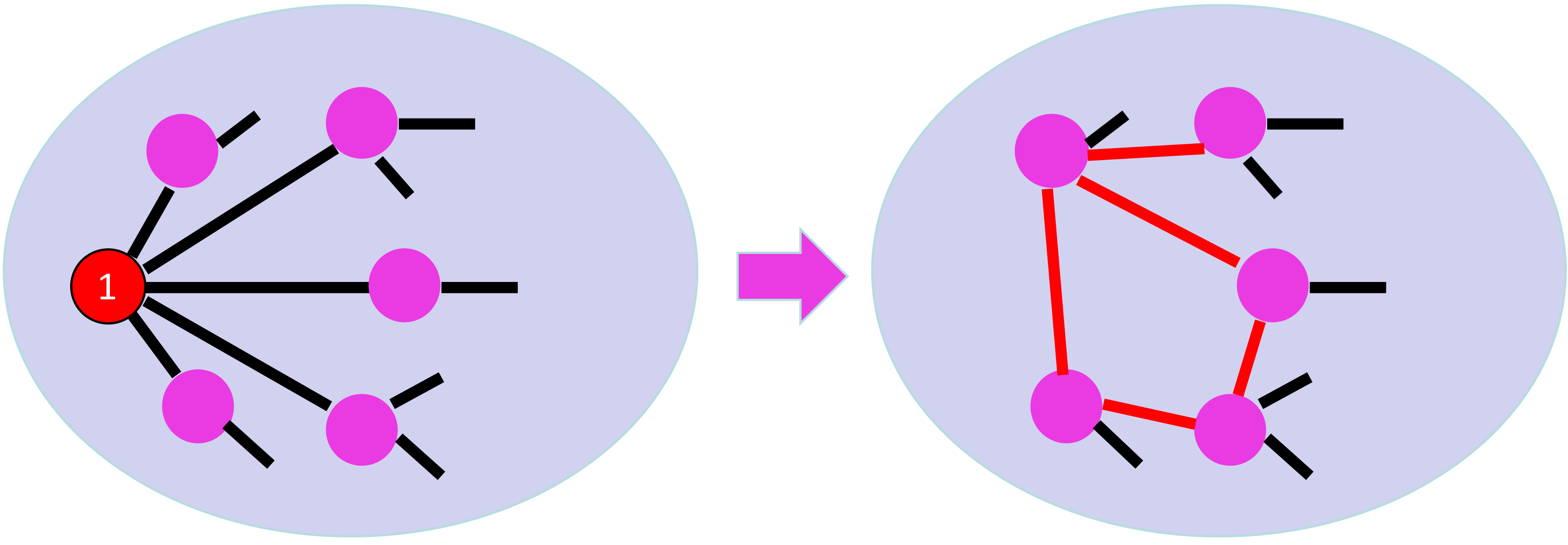
2. Sparsify that clique, without ever constructing it

Approximate Gaussian Elimination

(Kyng & Sachdeva '16)

1. When eliminate a node of degree d,

   add d edges at random between its neighbors, sampled with probability proportional to the weight of the edge to the eliminated node



Approximate Gaussian Elimination

(Kyng & Sachdeva '16)

0. Initialize by randomly permuting vertices, and making $O(\log^2 n)$ copies of every edge

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Total time is $O(m \log^3 n)$

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Total time is $O(m \log^3 n)$

Can be improved by sacrificing some simplicity

Approximate Gaussian Elimination

(Kyng & Sachdeva '16)

Analysis by Random Matrix Theory:

Write $U^T U$ as a sum of random matrices.

$$\mathbb{E}\left[U^T U\right] = L_G$$

Random permutation and copying
control the variances of the random matrices

Apply Matrix Freedman inequality (Tropp '11)

Recent Developments

Other families of linear systems

(Kyng, Lee, Peng, Sachdeva, S '16)

complex-weighted Laplacians $\begin{pmatrix} 1 & e^{i\theta} \\ e^{-i\theta} & 1 \end{pmatrix}$

connection Laplacians $\begin{pmatrix} I & Q \\ Q^T & I \end{pmatrix}$

Laplacians.jl

To learn more

My web page on:

Laplacian linear equations, sparsification, local graph clustering, low-stretch spanning trees, and so on.

My class notes from

"Graphs and Networks" and "Spectral Graph Theory"

$Lx = b$, by Nisheeth Vishnoi