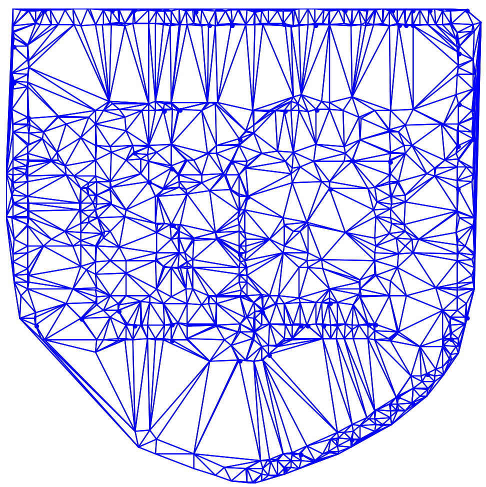
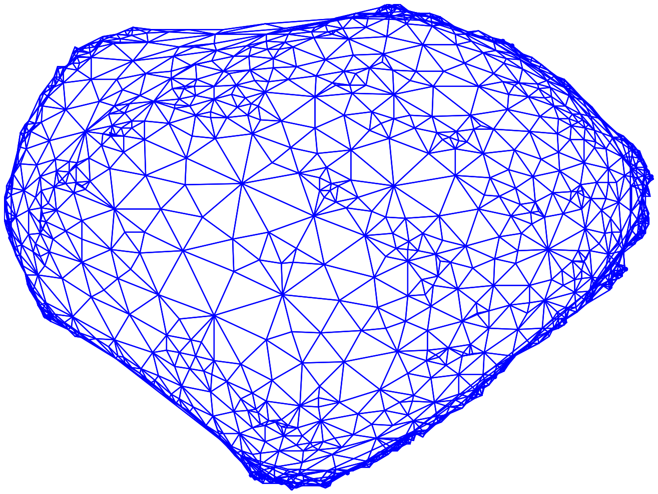
Spectral and Electrical Graph Theory





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Outline

Spectral Graph Theory: Understand graphs through eigenvectors and eigenvalues of associated matrices.

Electrical Graph Theory: Understand graphs through metaphor of resistor networks.

Heuristics
Algorithms
Theorems
Intuition

Spectral Graph Theory

Graph $G = (V, E)$

Matrix A

rows and cols
indexed by V

Eigenvalues $Av = \lambda v$

Eigenvectors $v : V \to \mathbb{R}$

Spectral Graph Theory

Graph $G = (V, E)$

Matrix A

rows and cols indexed by V

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

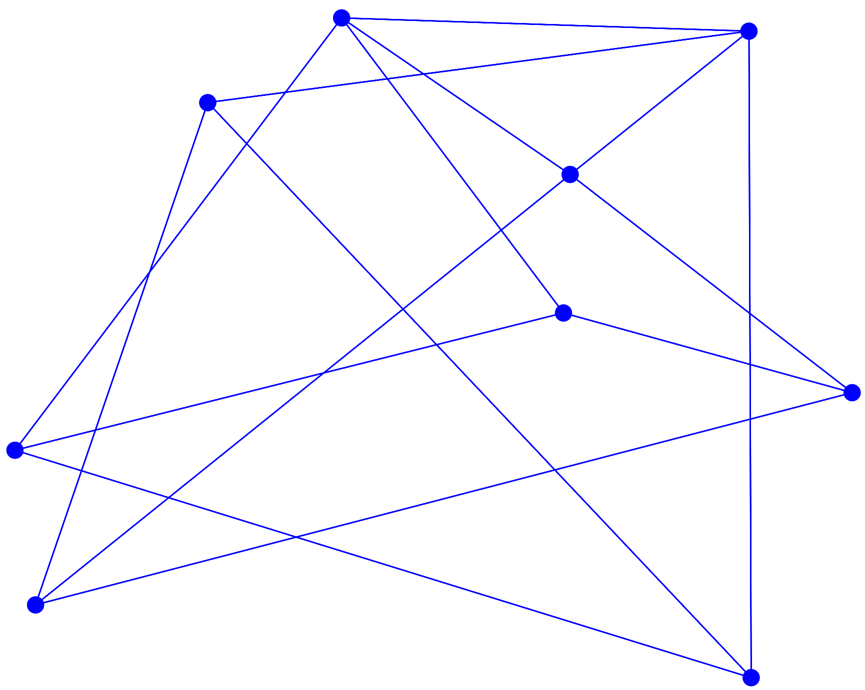
$$A(i, j) = 1 \text{ if } (i, j) \in E$$

Eigenvalues $Av = \lambda v$

Eigenvectors $v : V \to \mathbb{R}$

$-1 \quad -0.618 \quad 0.618 \quad 1$

Example: Graph Drawing by the Laplacian



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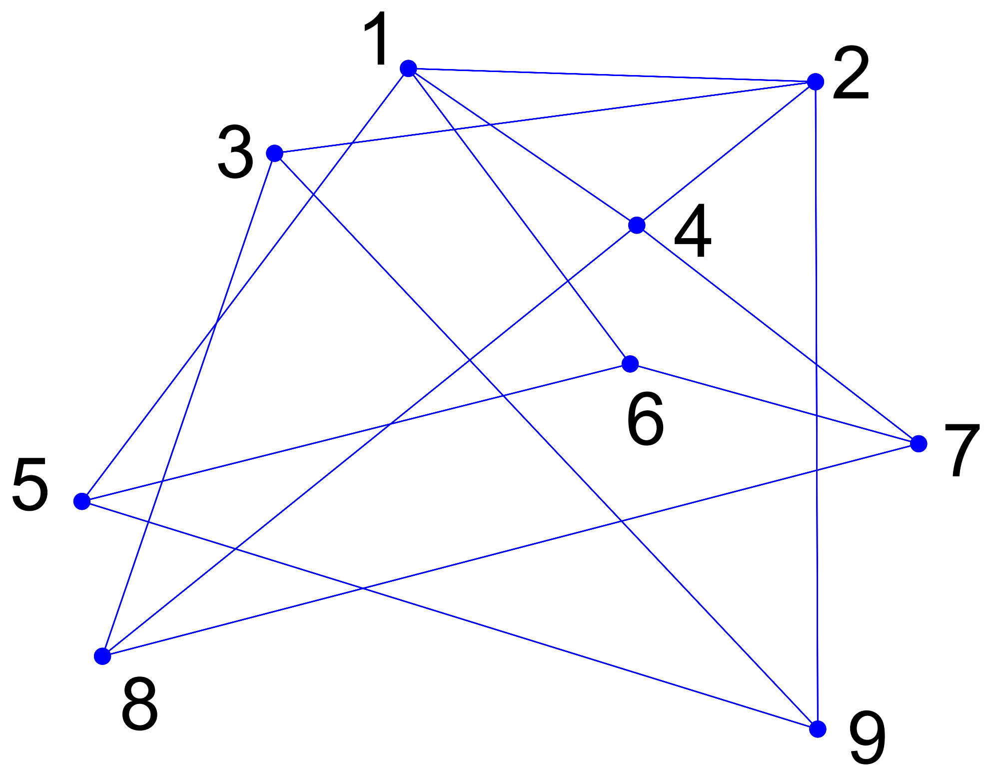
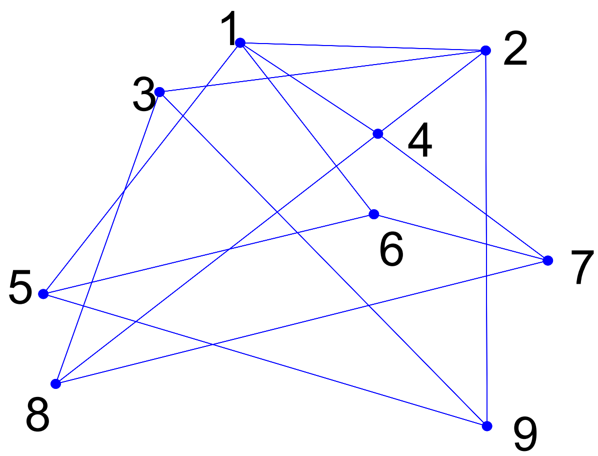
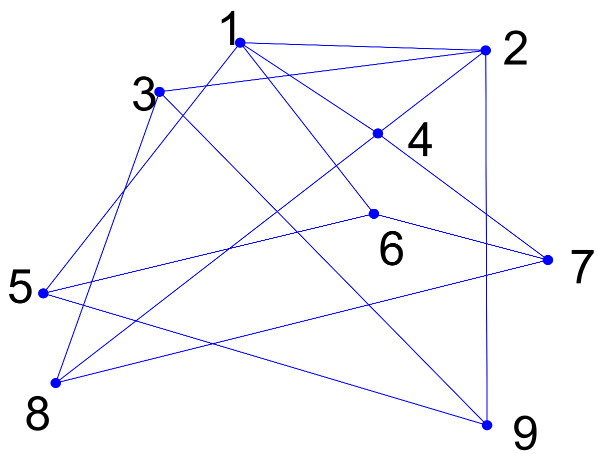


Example: Graph Drawing by the Laplacian



$$L(i,j) = \begin{cases} -1 & \text{if } (i,j) \in E \\ deg(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Example: Graph Drawing by the Laplacian

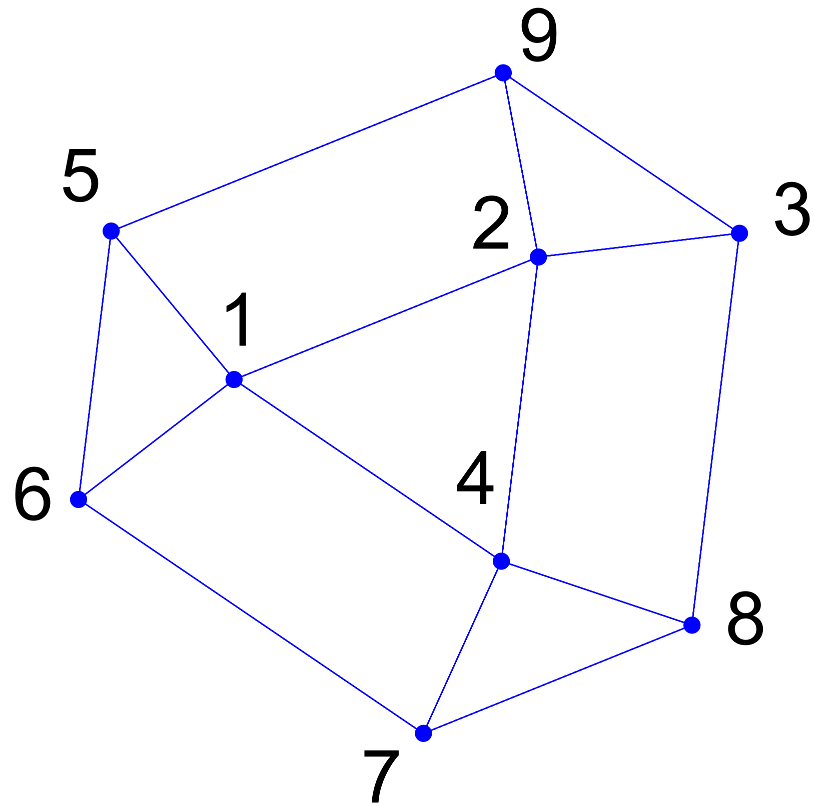


$$L(i,j) = \begin{cases} -1 & \text{if } (i,j) \in E \\ deg(i) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Eigenvalues 0, 1.53, 1.53, 3, 3.76, 3.76, 5, 5.7, 5.7

Let $x, y \in \mathbb{R}^V$ span eigenspace of eigenvalue 1.53

Example: Graph Drawing by the Laplacian



Plot vertex i at $(x(i), y(i))$

Draw edges as straight lines

Laplacian: natural quadratic form on graphs

$$x^T L x = \sum_{(i,j)\in E} (x(i) - x(j))^2$$

$$L = D - A$$ where D is diagonal matrix of degrees

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Laplacian: fast facts

$$x^T L x = \sum_{(i,j)\in E} (x(i) - x(j))^2$$

$\boldsymbol{L1 = 0}$ zero is an eigenvalue

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

Connected if and only if $\lambda_2 > 0$

Fiedler ('73) called λ_2
"algebraic connectivity of a graph"
The further from 0, the more connected.

Drawing a graph in the line (Hall '70)

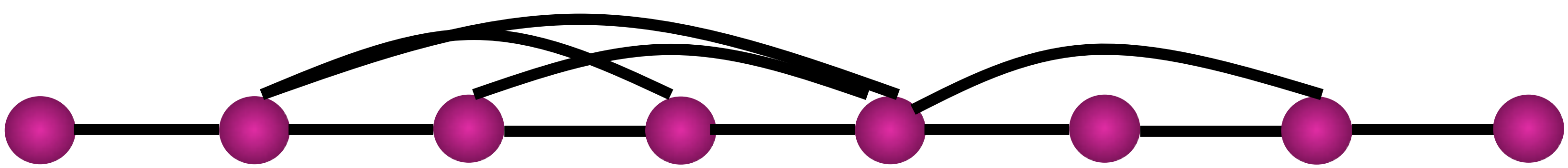
map $V \longrightarrow \mathbb{R}$

minimize $\displaystyle\sum_{(i,j)\in E} (x(i) - x(j))^2 = x^T L x$

trivial solution: $x = \mathbf{1}$ So, require $x \perp \mathbf{1}, \|x\| = 1$

Solution $x = v_2$

Atkins, Boman, Hendrickson '97:
Gives correct drawing for graphs like



Courant-Fischer definition of eigvals/vecs

$$\lambda_1 = \min_{x \neq 0} \frac{x^T L x}{x^T x} \qquad v_1 = \arg\min_{x \neq 0} \frac{x^T L x}{x^T x}$$

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$$\lambda_2 = \min_{x \perp v_1} \frac{x^T L x}{x^T x} \qquad v_2 = \arg\min_{x \perp v_1} \frac{x^T L x}{x^T x}$$

(here $v_1 = \mathbf{1}$)

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(here $v_1 = \mathbf{1}$)

$$\boxed{\begin{aligned} \lambda_k &= \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T L x}{x^T x} \\ v_k &= \arg \min_{x \perp v_1, \ldots, v_{k-1}} \frac{x^T L x}{x^T x} \end{aligned}}$$

Drawing a graph in the plane (Hall '70)

map $V \longrightarrow \mathbb{R}^2$ $\vec{x}(i) \in \mathbb{R}^2$

minimize $\sum_{(i,j)\in E} (\mathrm{dist}(\vec{x}(i), \vec{x}(j))^2$

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So, require $\vec{x}_1, \vec{x}_2 \perp \mathbf{1}$

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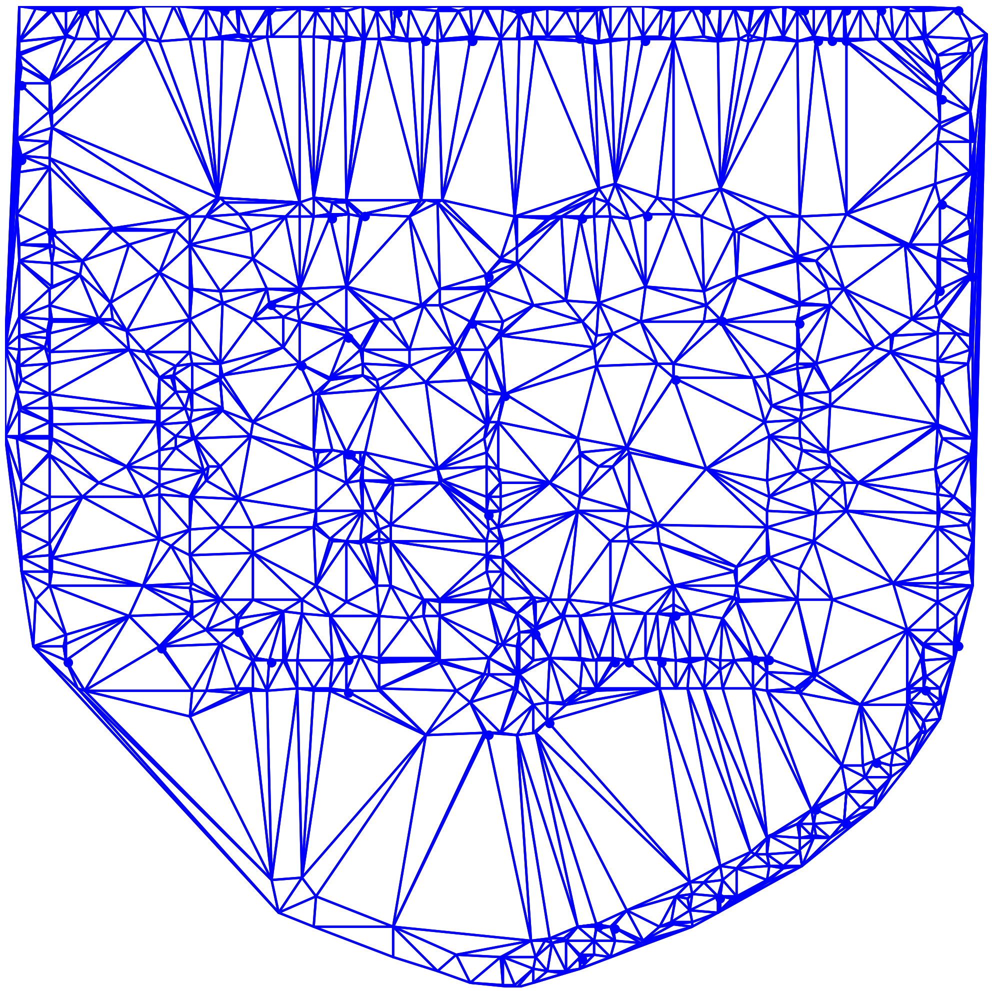
So, require $\vec{x}_1, \vec{x}_2 \perp \mathbf{1}$

diagonal solution: $\vec{x}(i) = (v_2(i), v_2(i))$

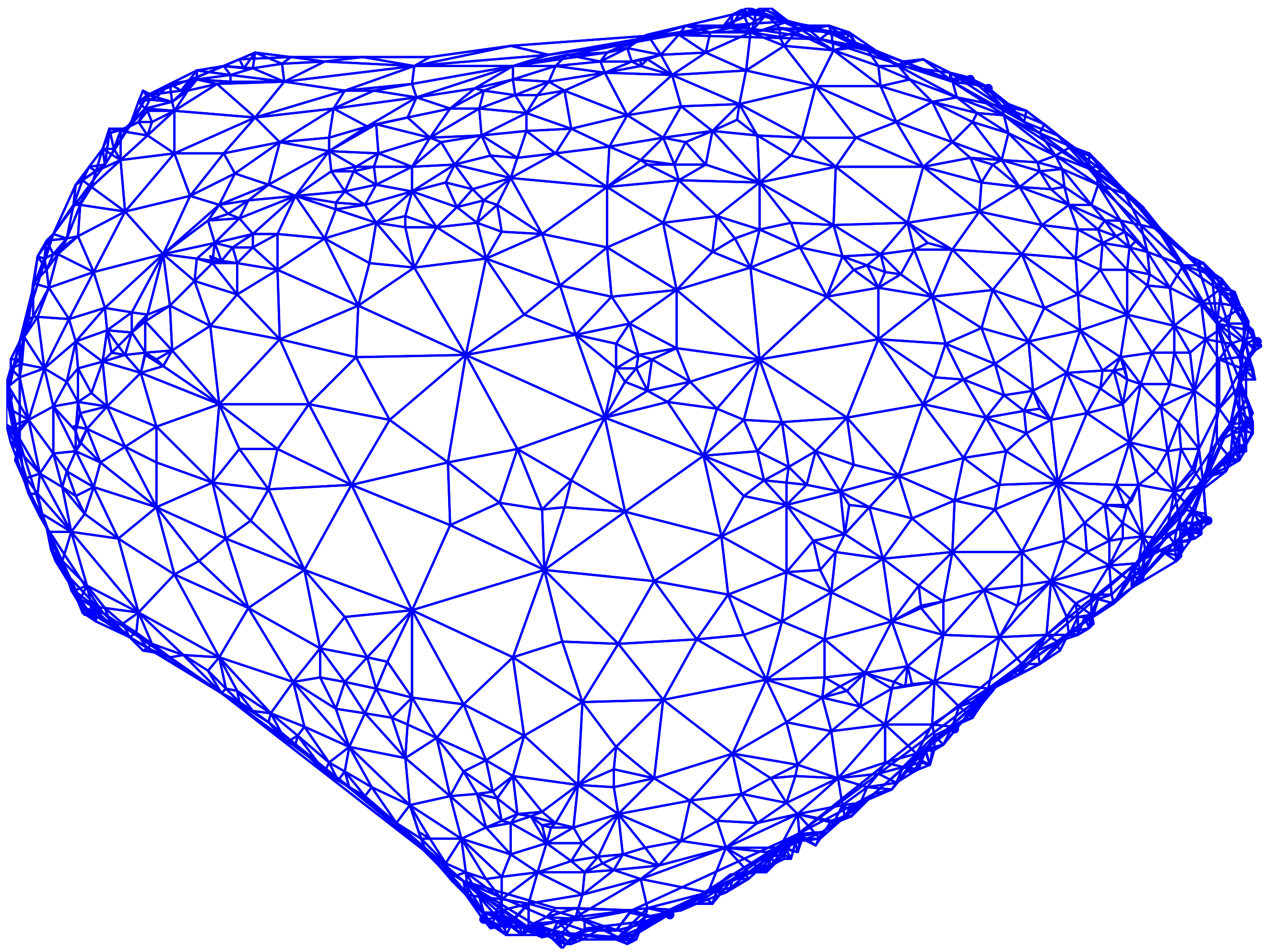
So, require $\vec{x}_1 \perp \vec{x}_2$

Solution $\vec{x}(i) = (v_2(i), v_3(i))$ up to rotation

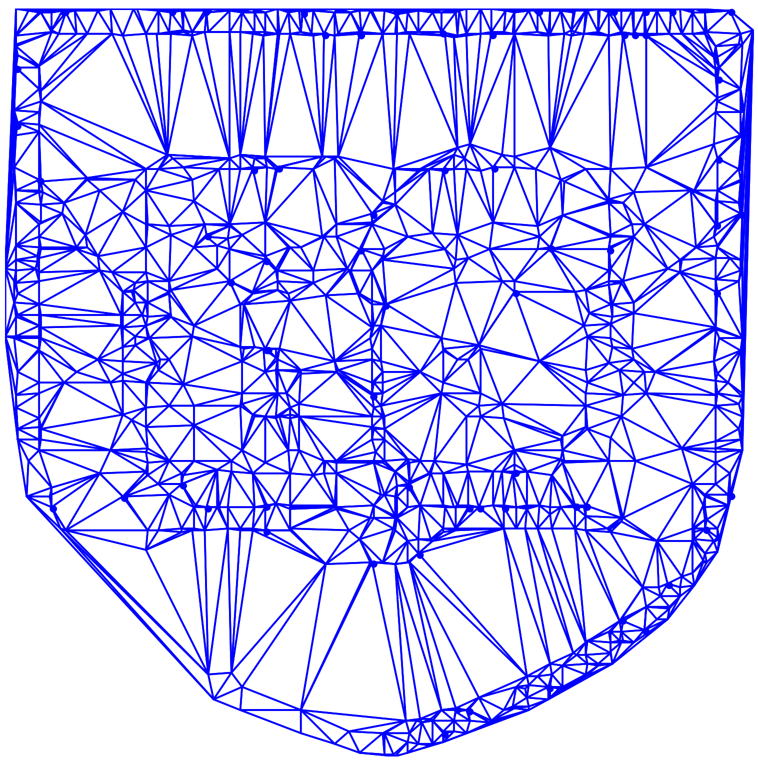
A Graph

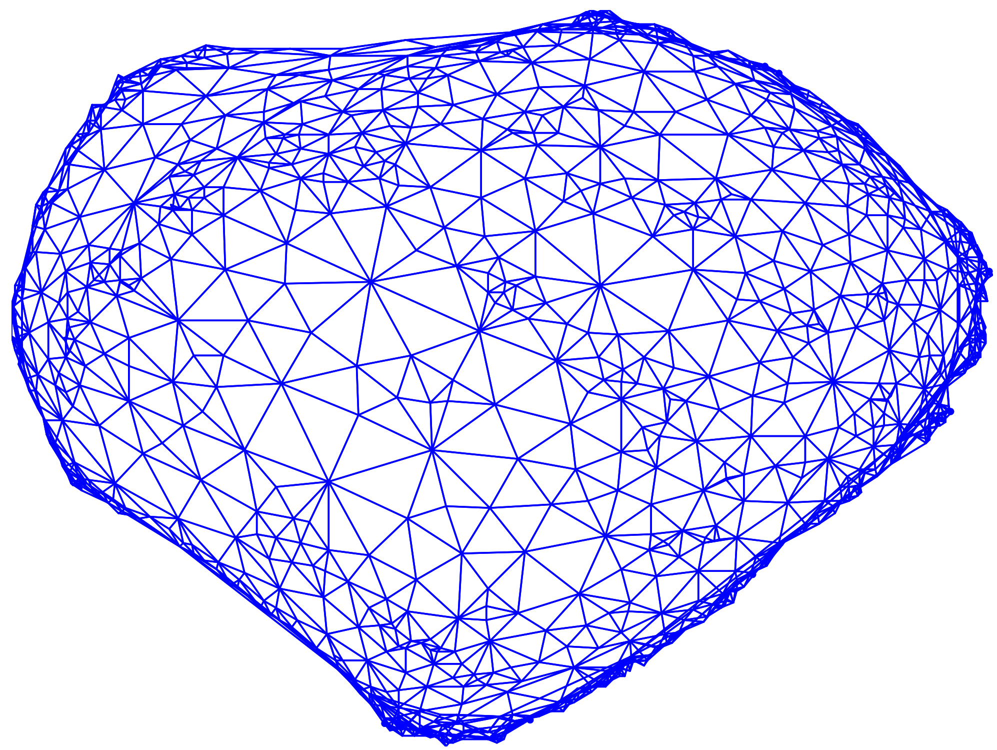


Drawing of the graph using v_2, v_3

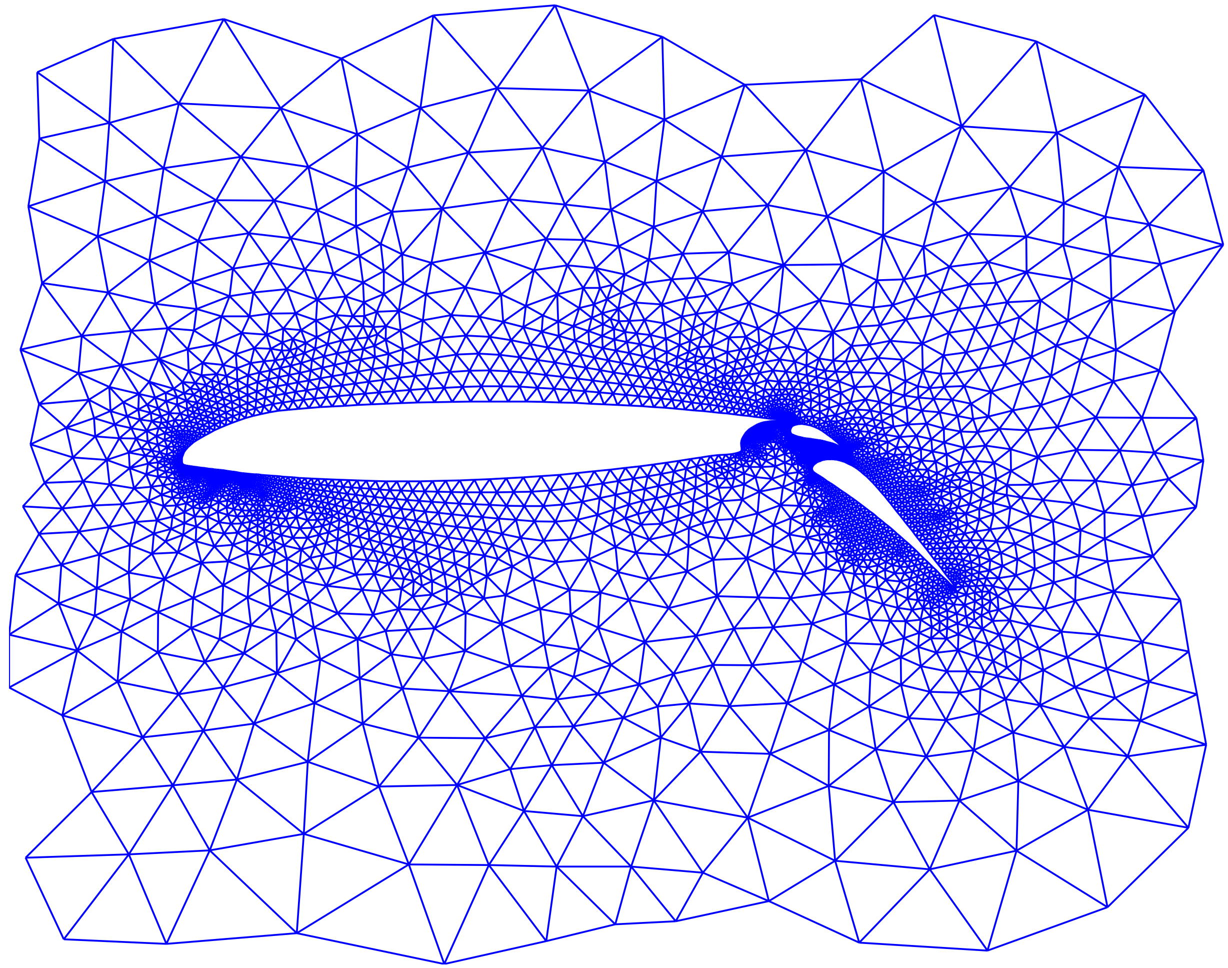


Plot vertex i at $(v_2(i), v_3(i))$

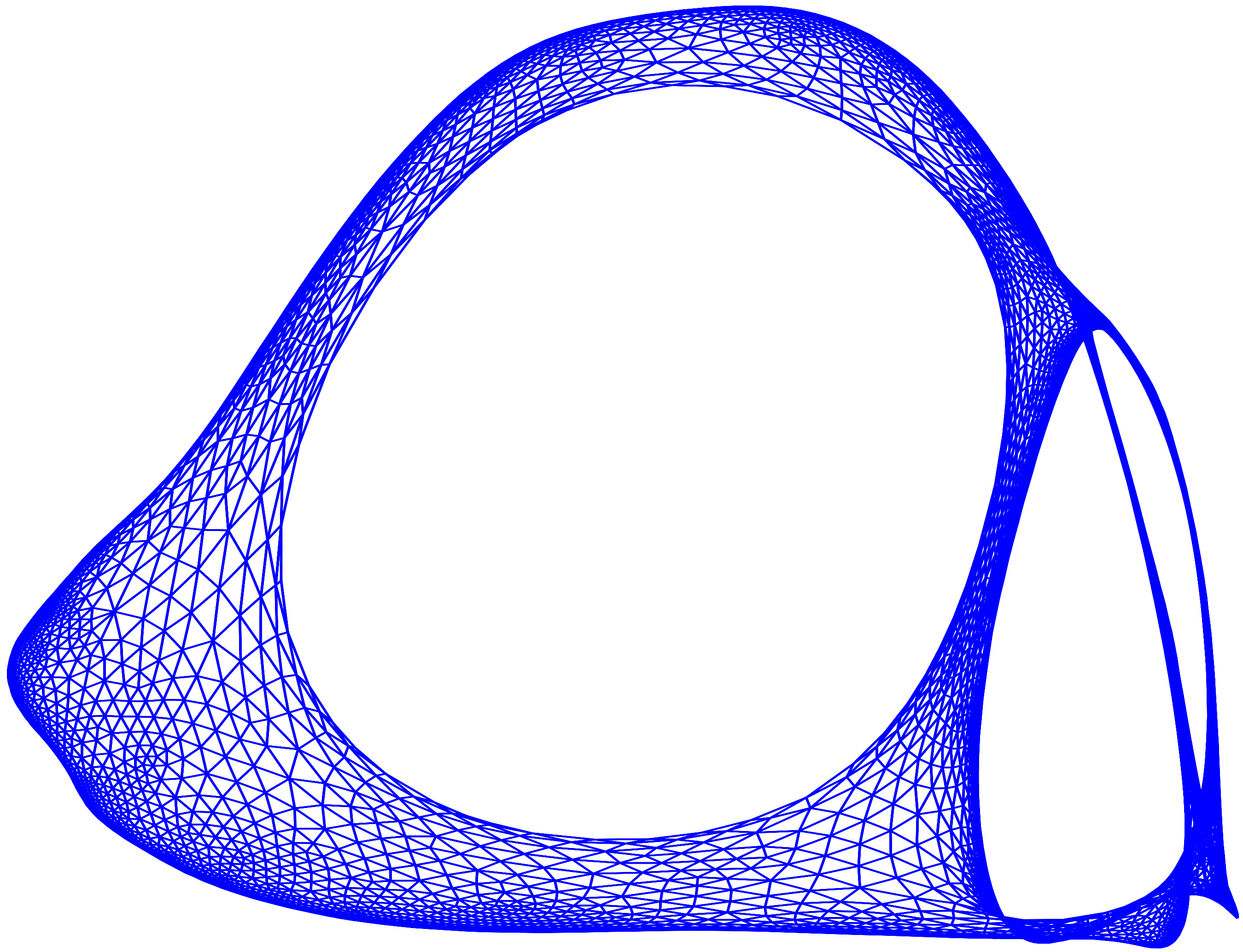




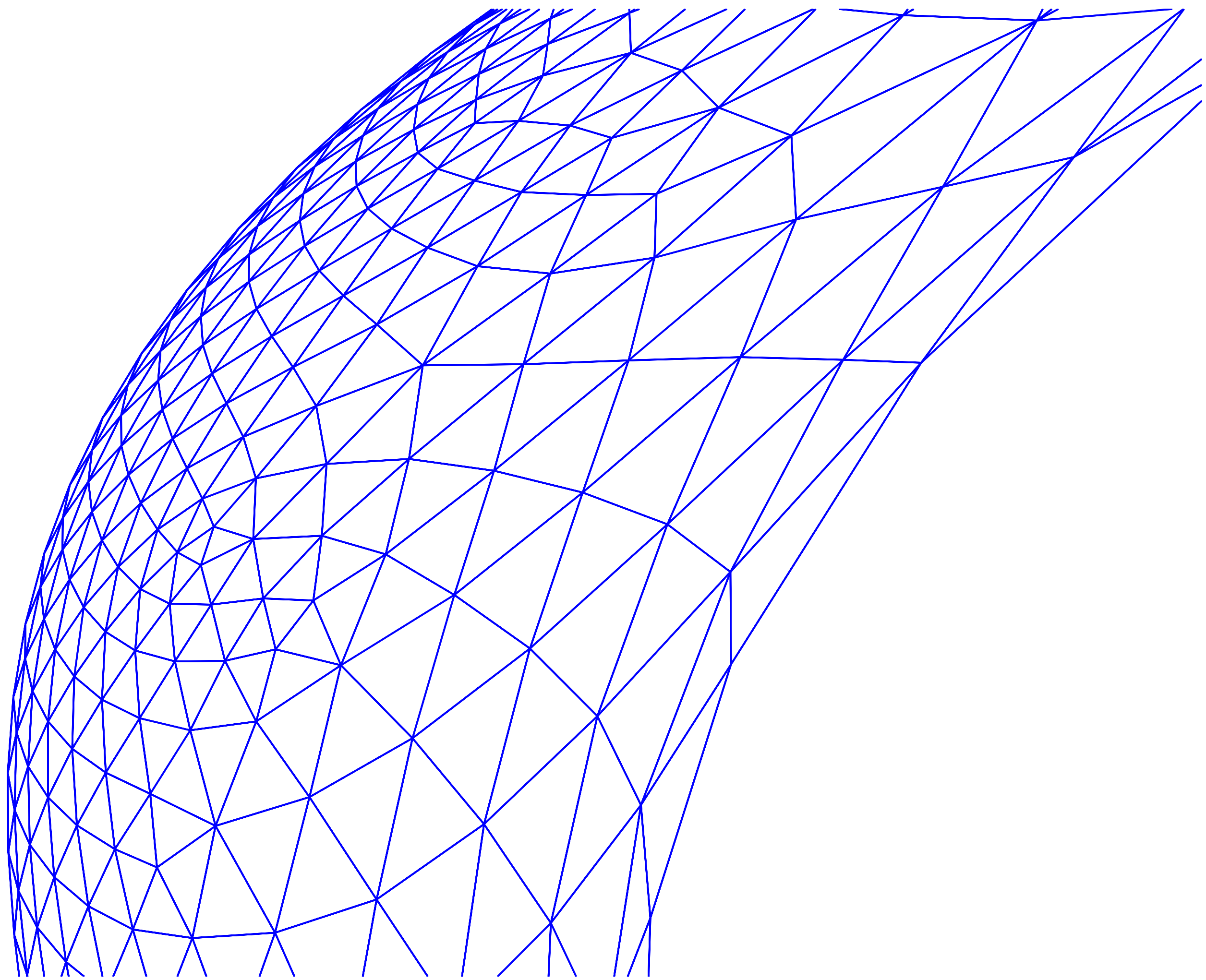
The Airfoil Graph, original coordinates



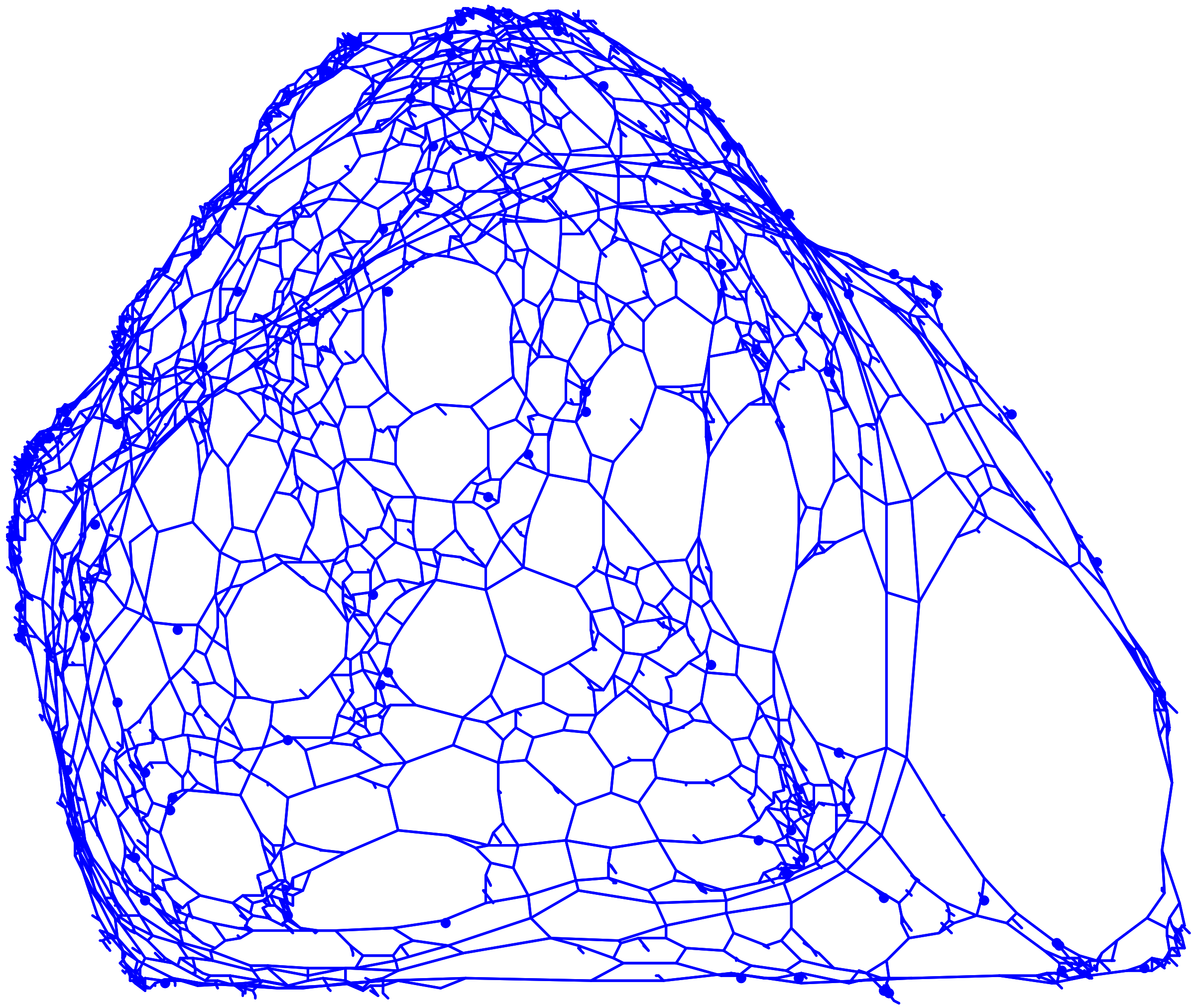
The Airfoil Graph, spectral coordinates



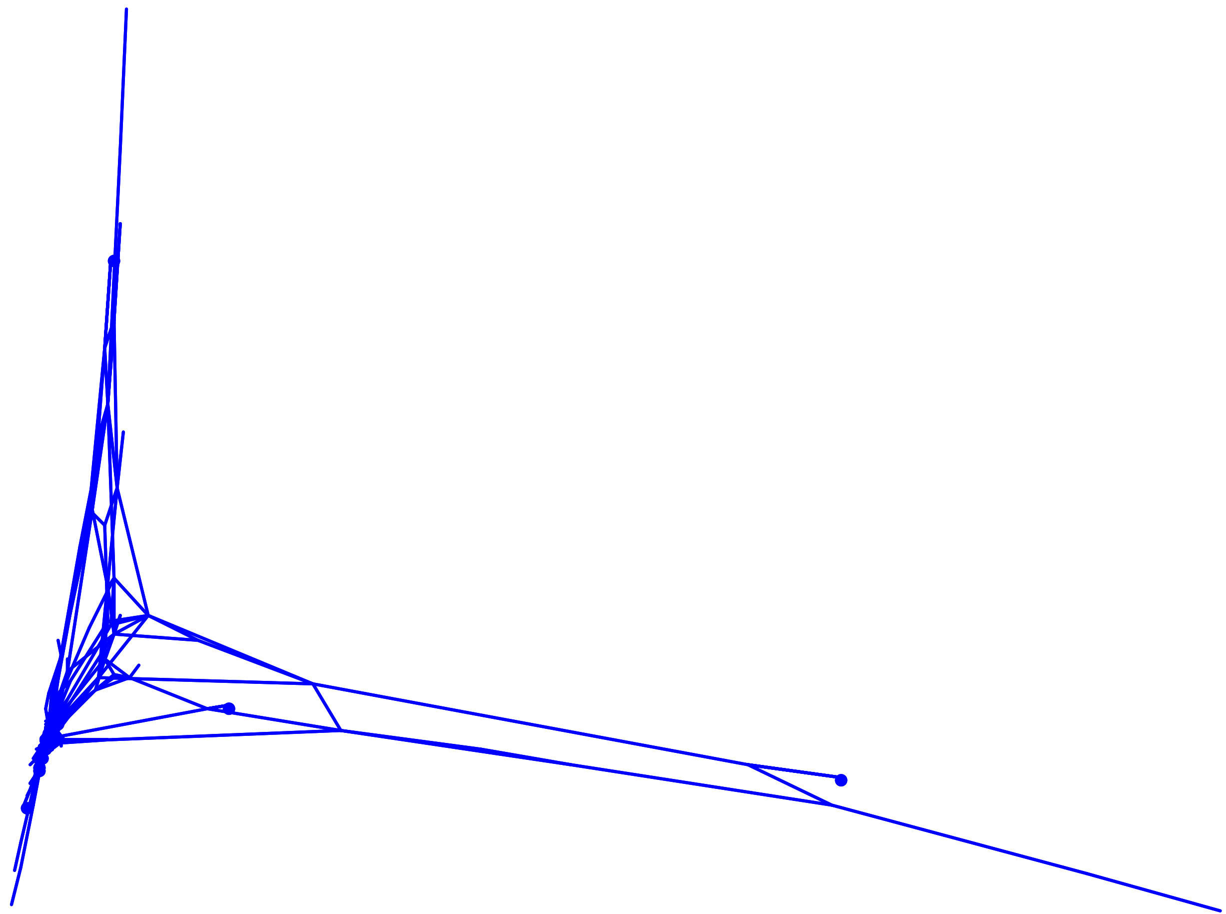
The Airfoil Graph, spectral coordinates



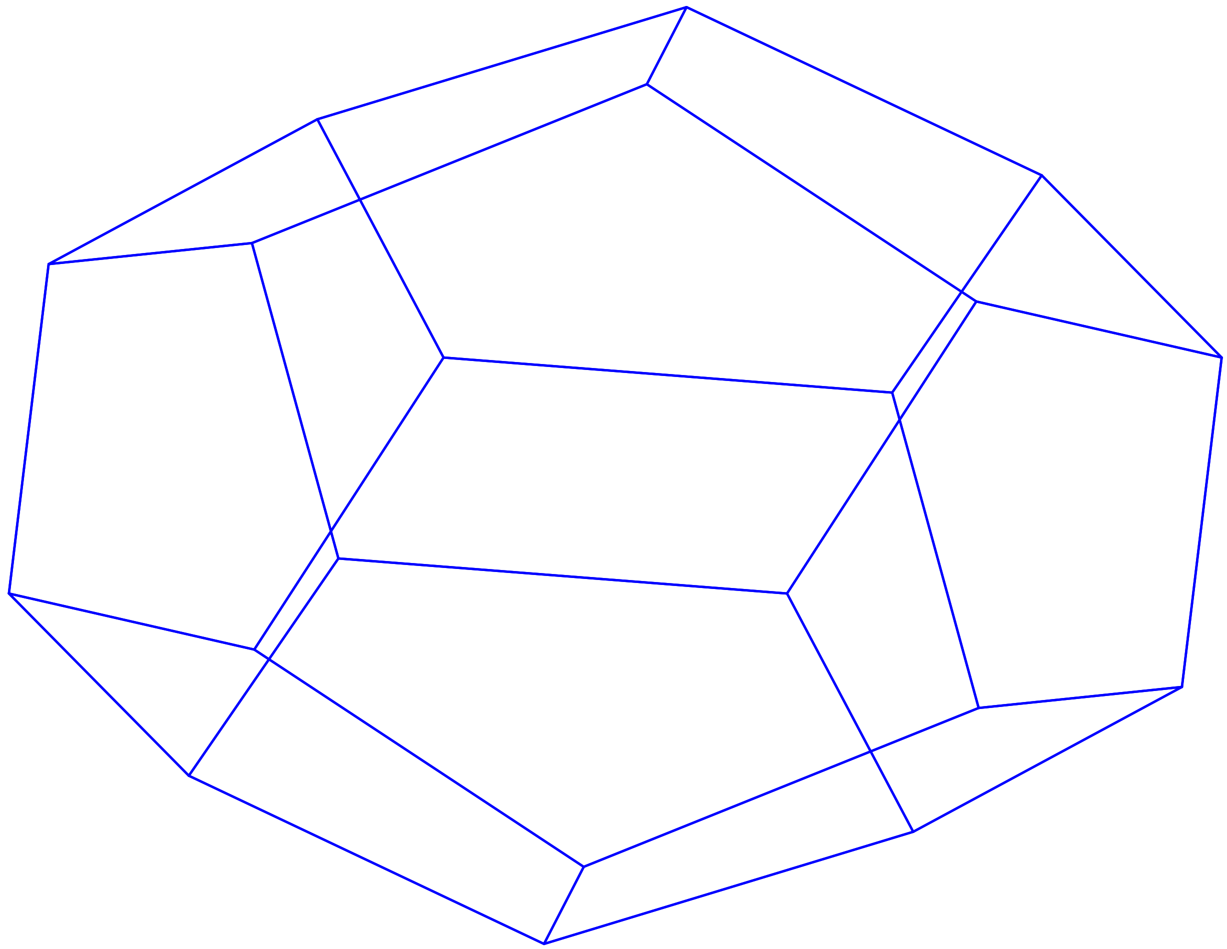
Spectral drawing of Streets in Rome



Spectral drawing of Erdos graph: edge between co-authors of papers



Dodecahedron



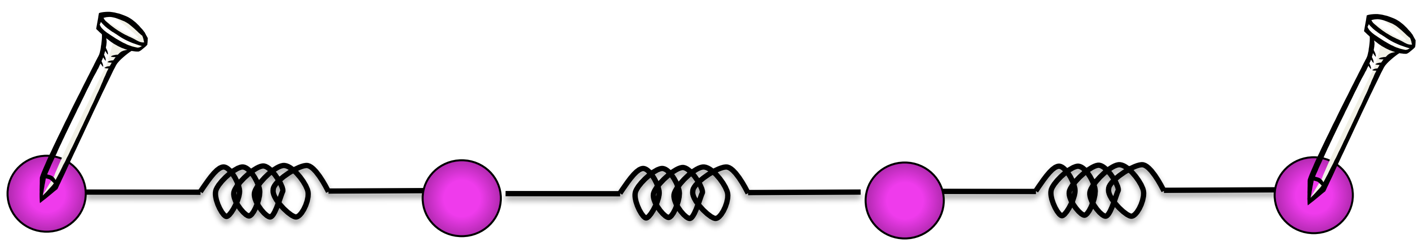
Best embedded by first three eigenvectors

Intuition: Graphs as Spring Networks

edges -> ideal linear springs
weights -> spring constants (k)

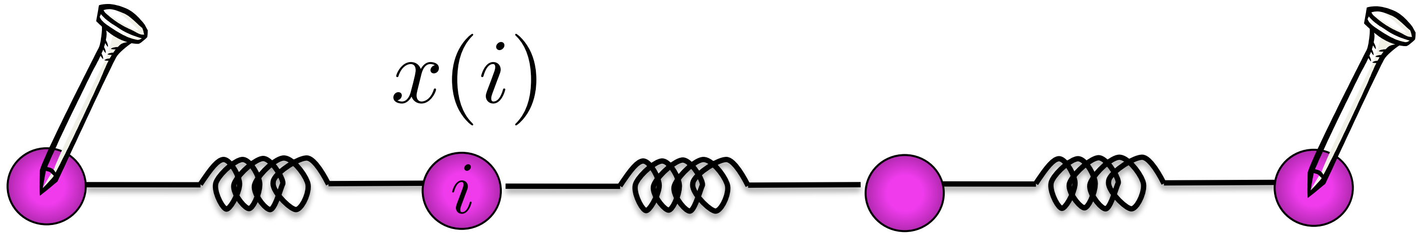
Physics: when stretched to length x, force is kx
potential energy is $kx^2/2$

Nail down some vertices, let rest settle



Intuition: Graphs as Spring Networks

Nail down some vertices, let rest settle



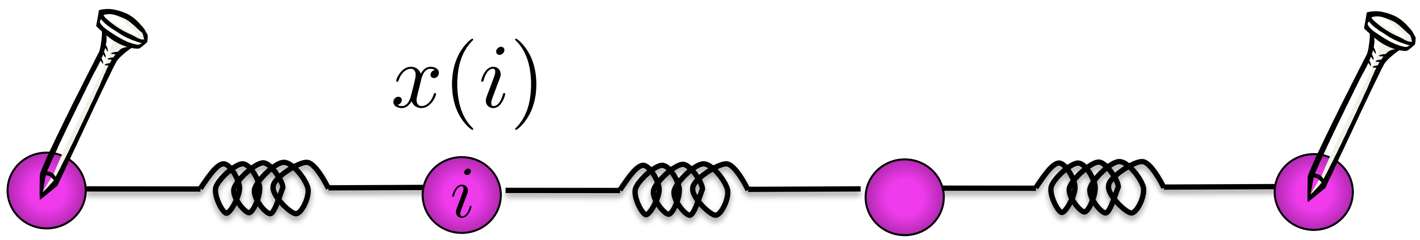
Physics: minimizes total potential energy

$$\sum_{(i,j)\in E} (x(i) - x(j))^2 = x^T L x$$

subject to boundary constraints (nails)

Intuition: Graphs as Spring Networks

Nail down some vertices, let rest settle

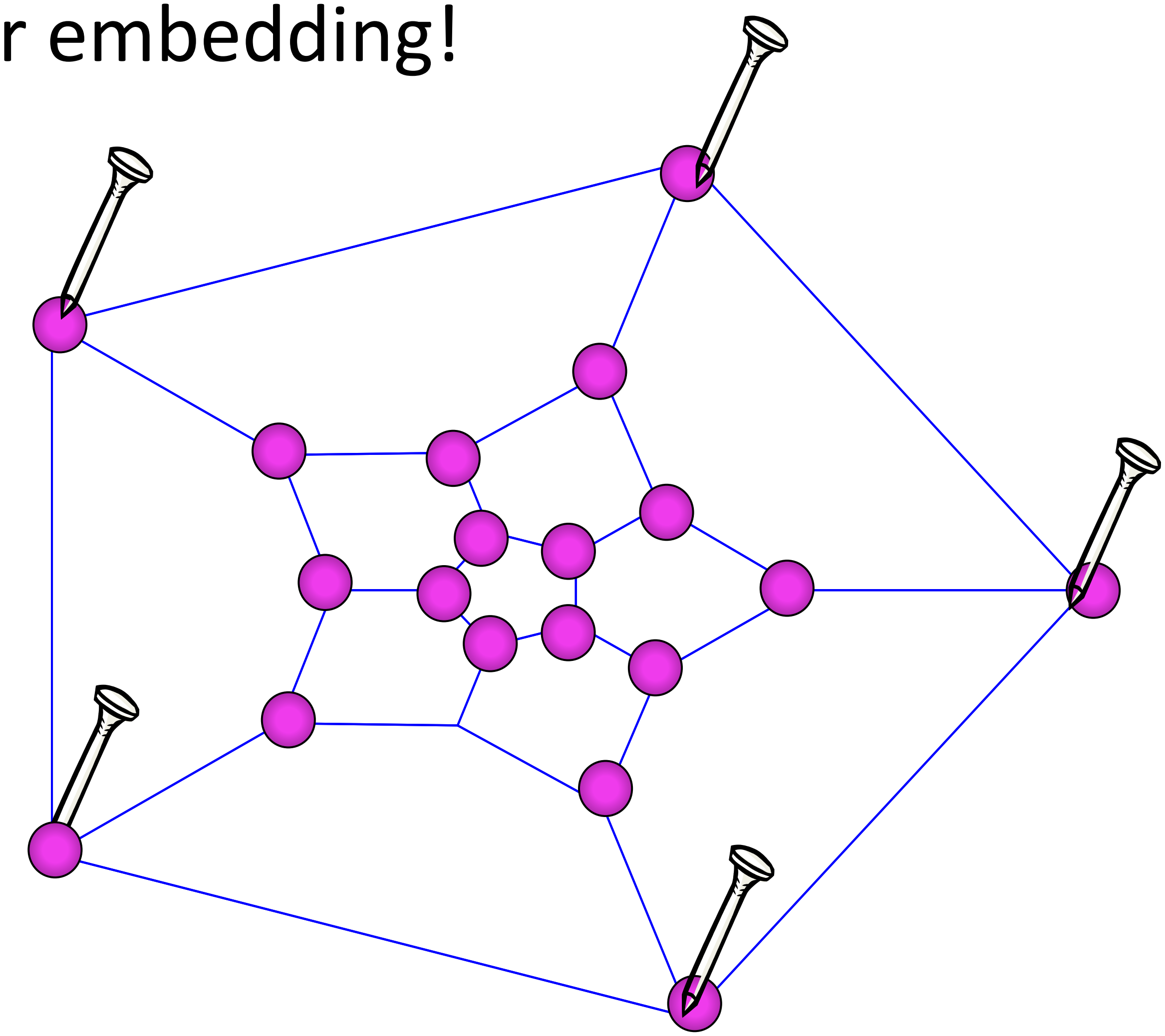


Physics: energy minimized when
non-fixed vertices are averages of neighbors

$$\vec{x}(i) = \frac{1}{d_i} \sum_{(i,j) \in E} \vec{x}(j)$$

Tutte's Theorem '63

If nail down a face of a planar 3-connected graph, get a planar embedding!



Spectral graph drawing: Tutte justification

Condition for eigenvector $Lx = \lambda x$

Gives $x(i) = \dfrac{1}{d_i - \lambda} \sum_{(i,j) \in E} x(j)$ for all i

λ small says $x(i)$ near average of neighbors

Spectral graph drawing: Tutte justification

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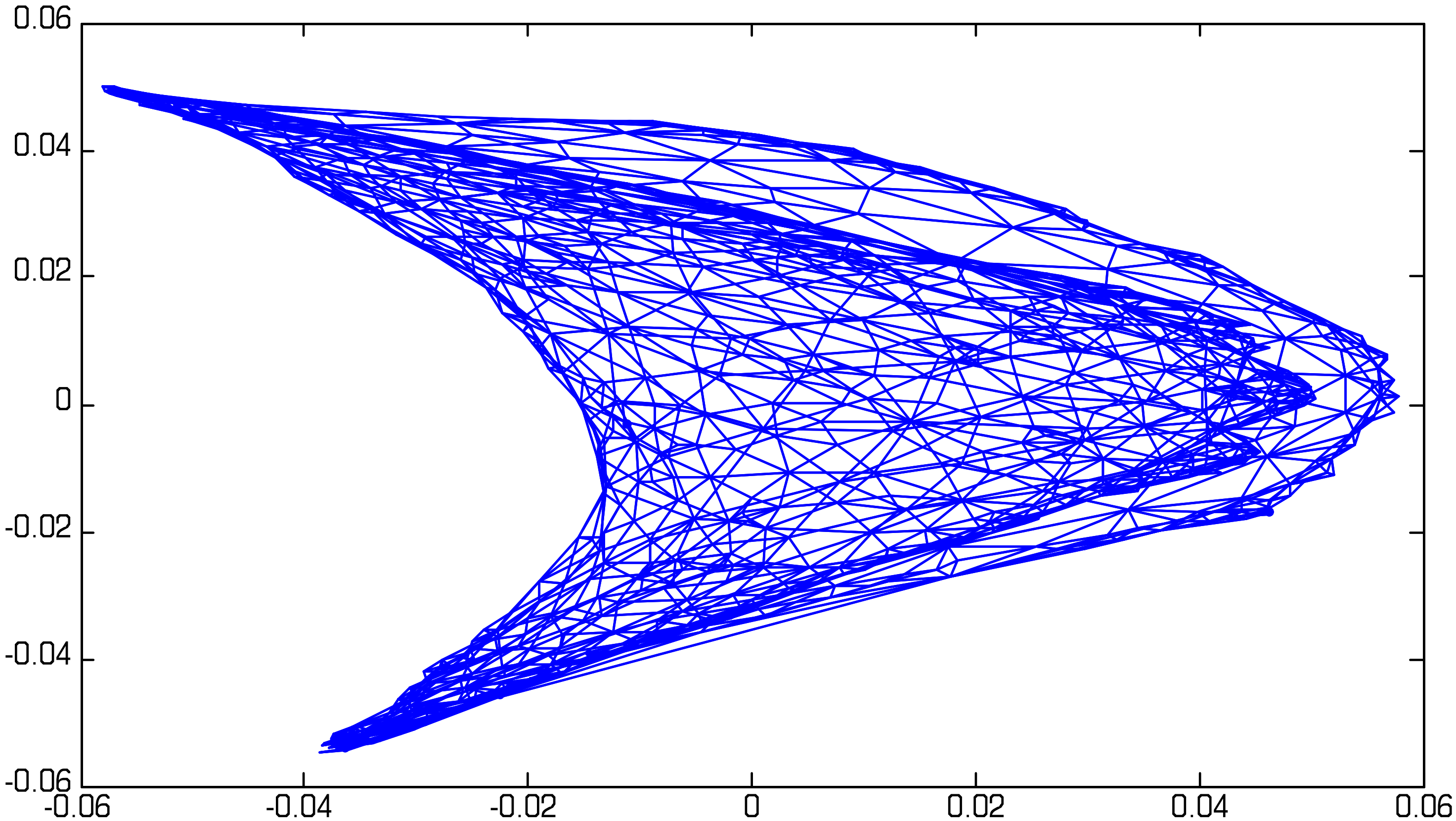
λ small says $x(i)$ near average of neighbors

For planar graphs:

$\lambda_2 \leq 8d/n$ [S-Teng '96]

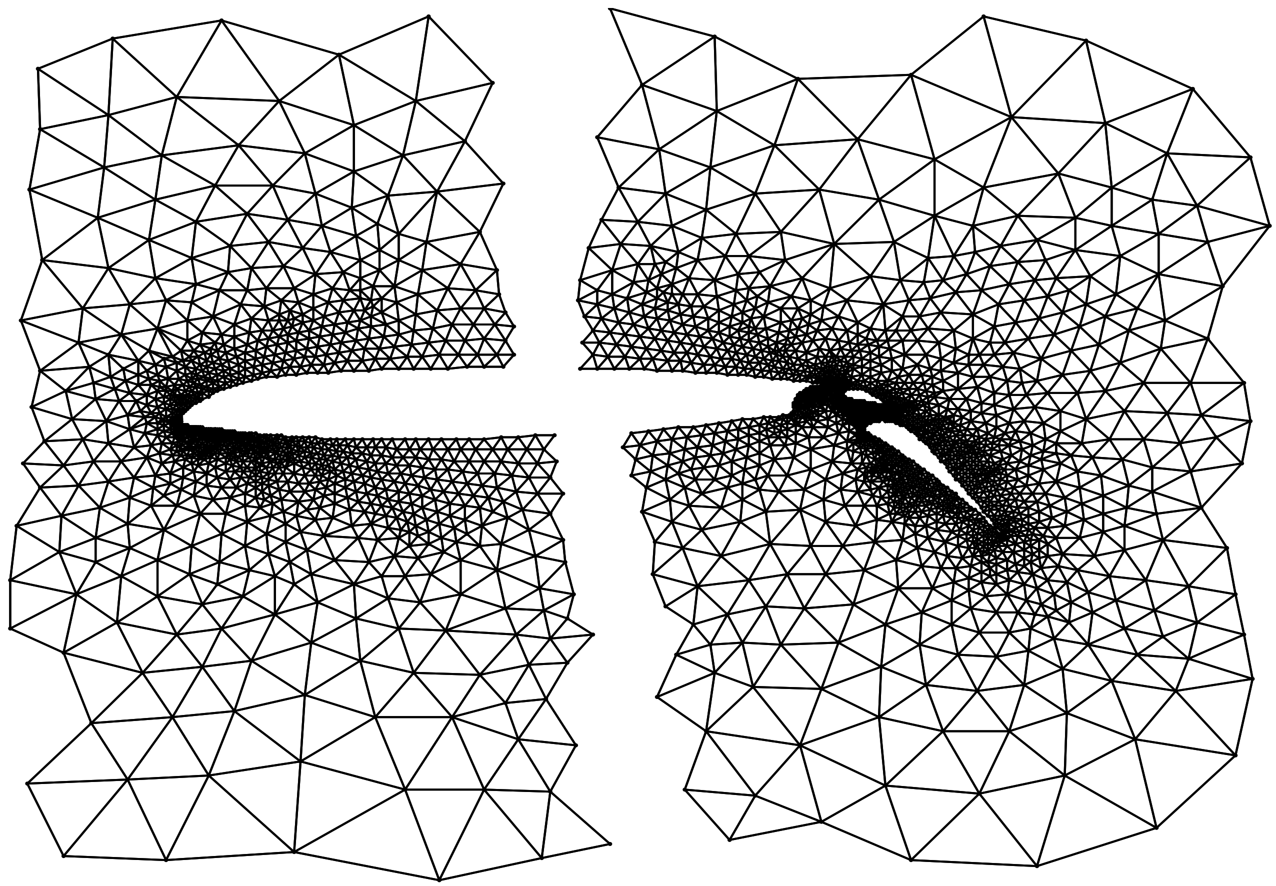
$\lambda_3 \leq O(d/n)$ [Kelner-Lee-Price-Teng '09]

Small eigenvalues are not enough



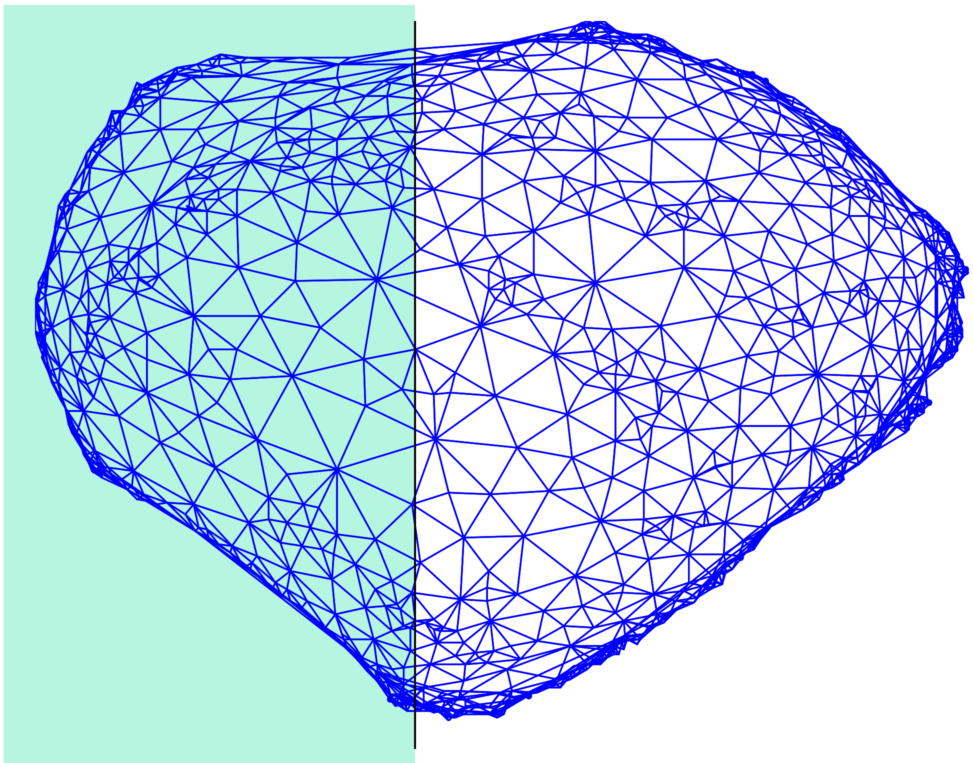
Plot vertex i at $(v_3(i), v_4(i))$

Graph Partitioning



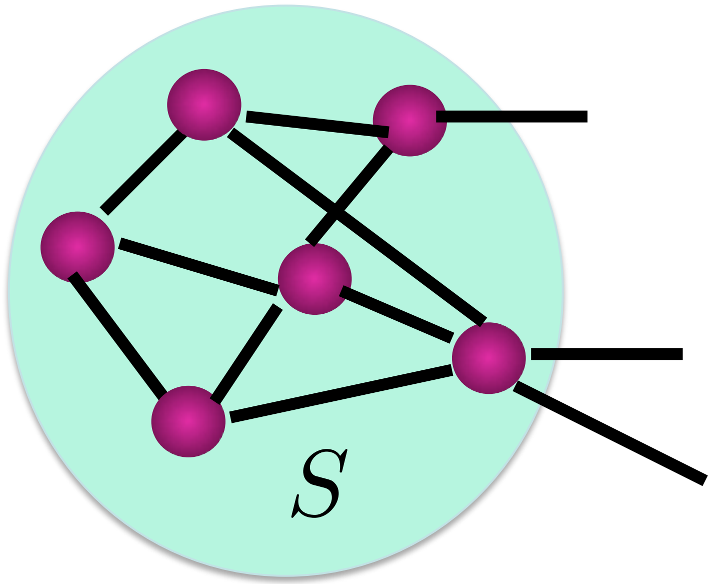
Spectral Graph Partitioning

[Donath-Hoffman '72, Barnes '82, Hagen-Kahng '92]



$$S = \{i : v_2(i) \leq t\} \text{ for some } t$$

Measuring Partition Quality: Conductance



$$\Phi(S) = \frac{\#\text{ edges leaving } S}{\text{sum of degrees in } S}$$

For $\deg(S) \leq \deg(V)/2$

Spectral Image Segmentation (Shi-Malik ‘00)

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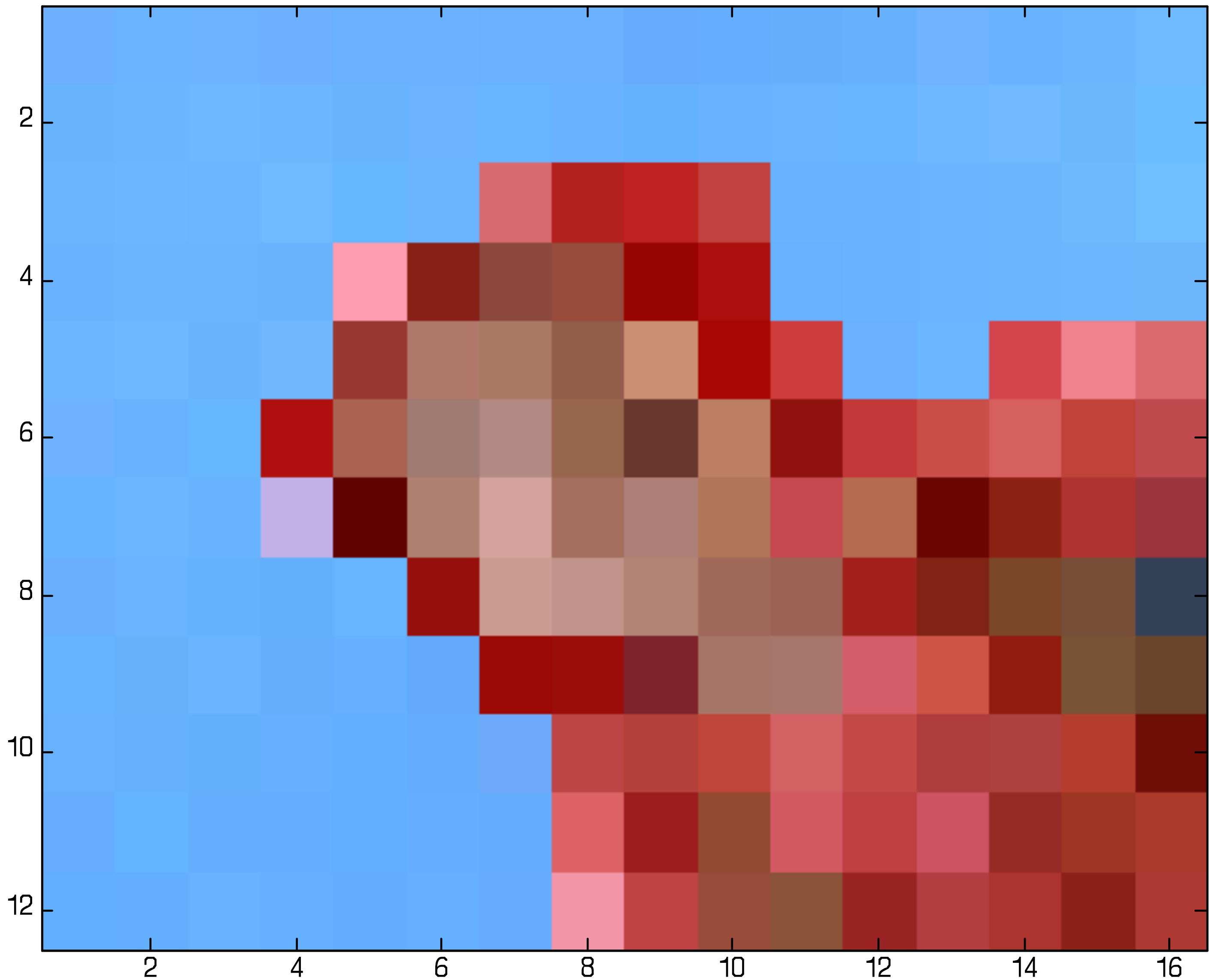

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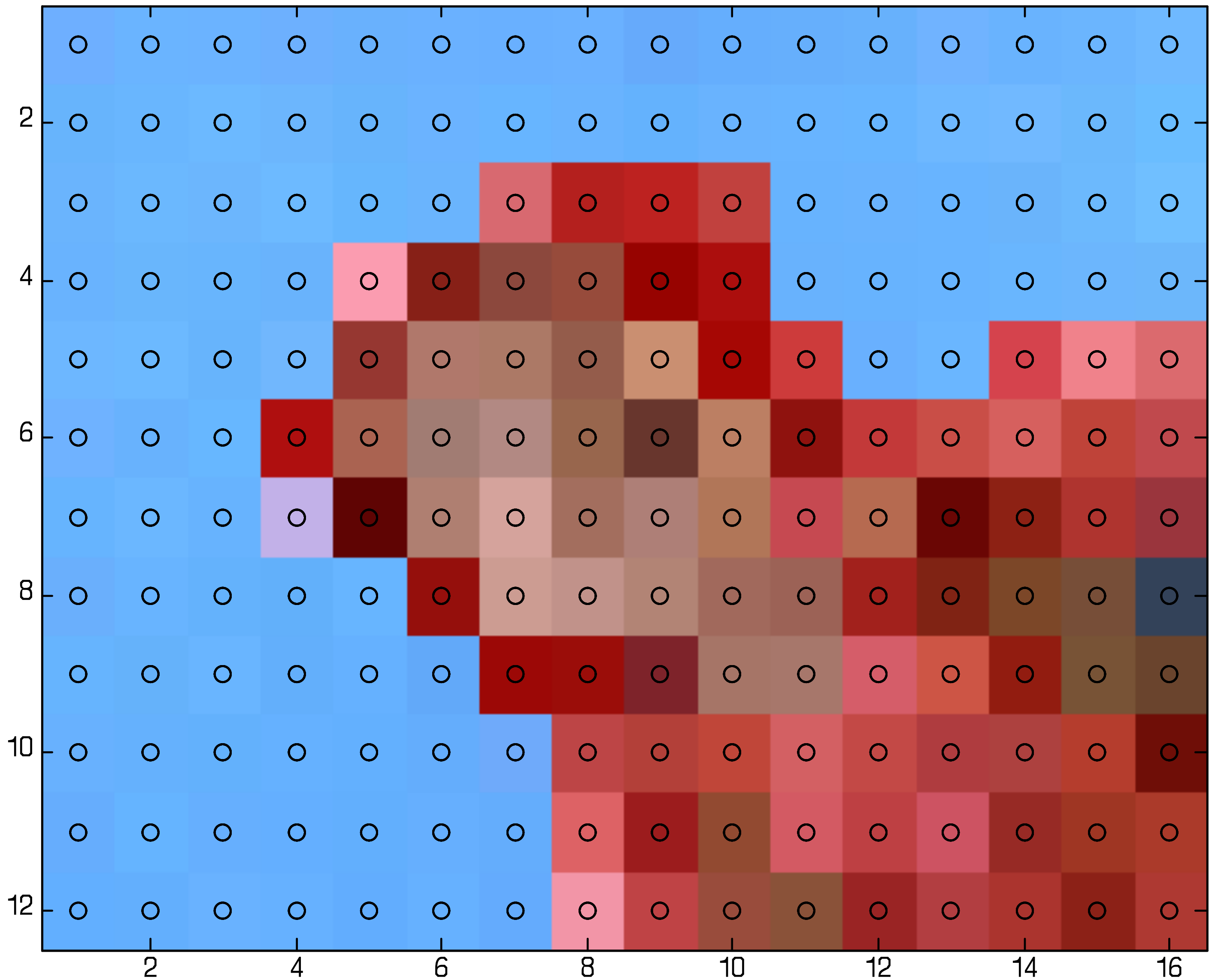

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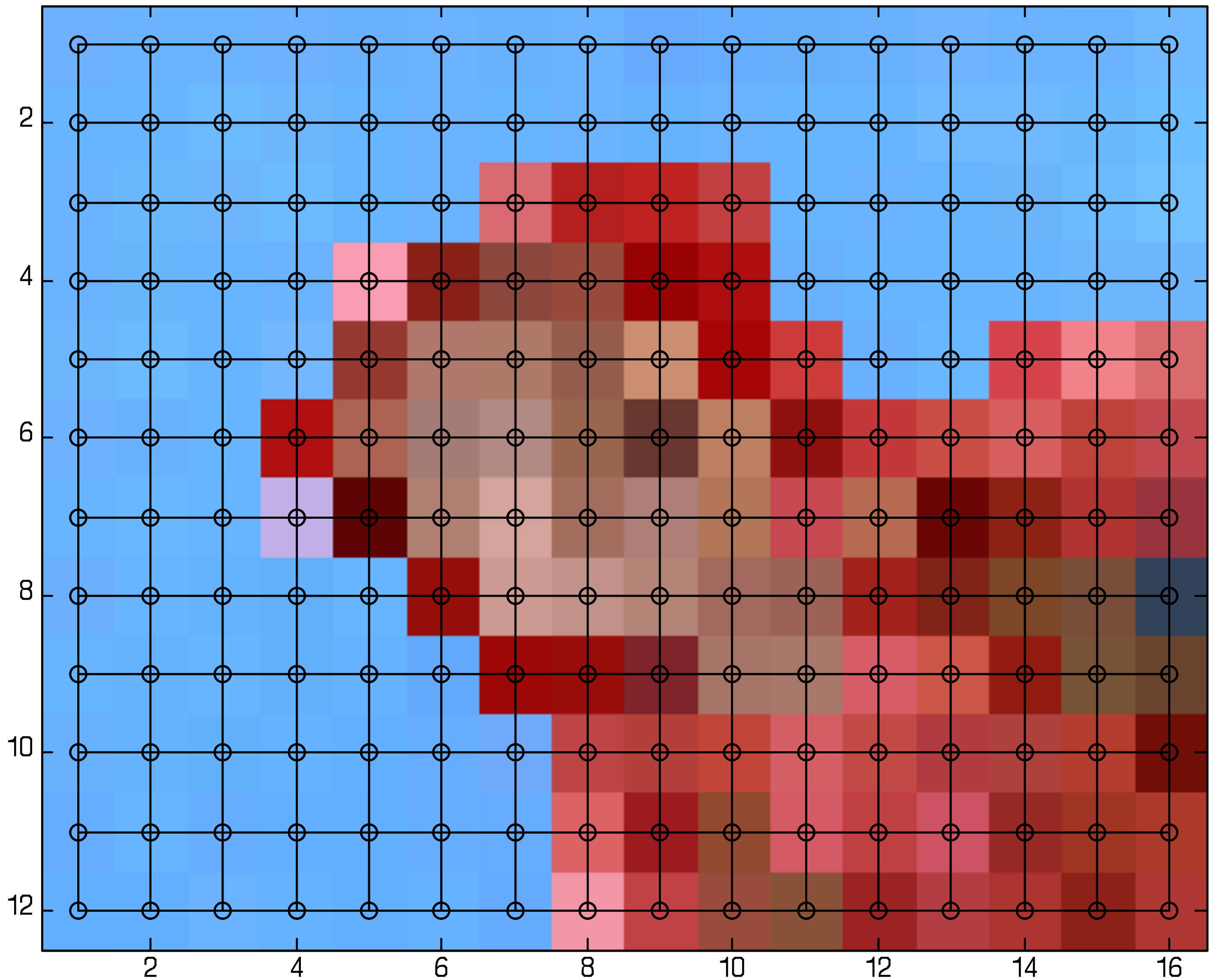

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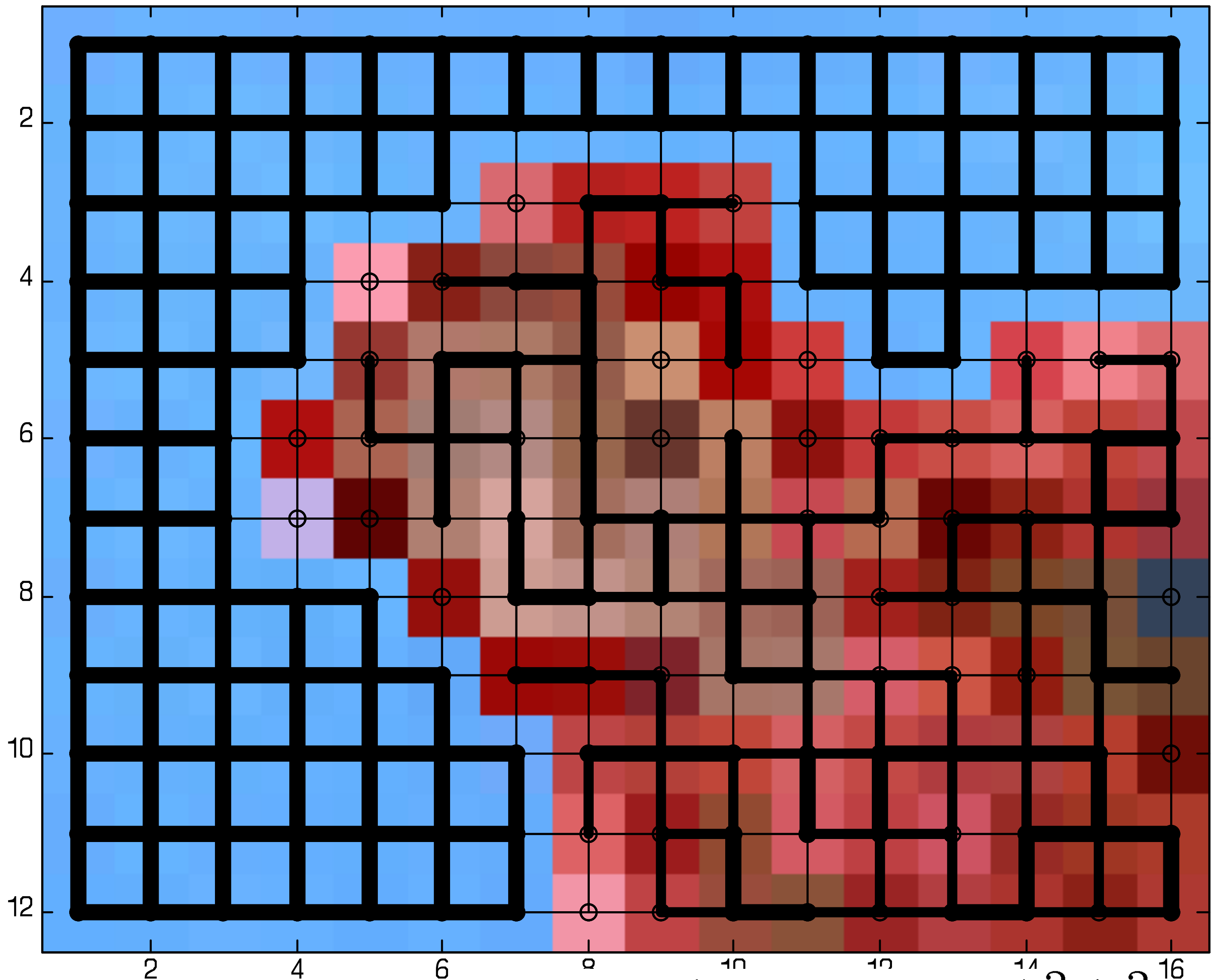
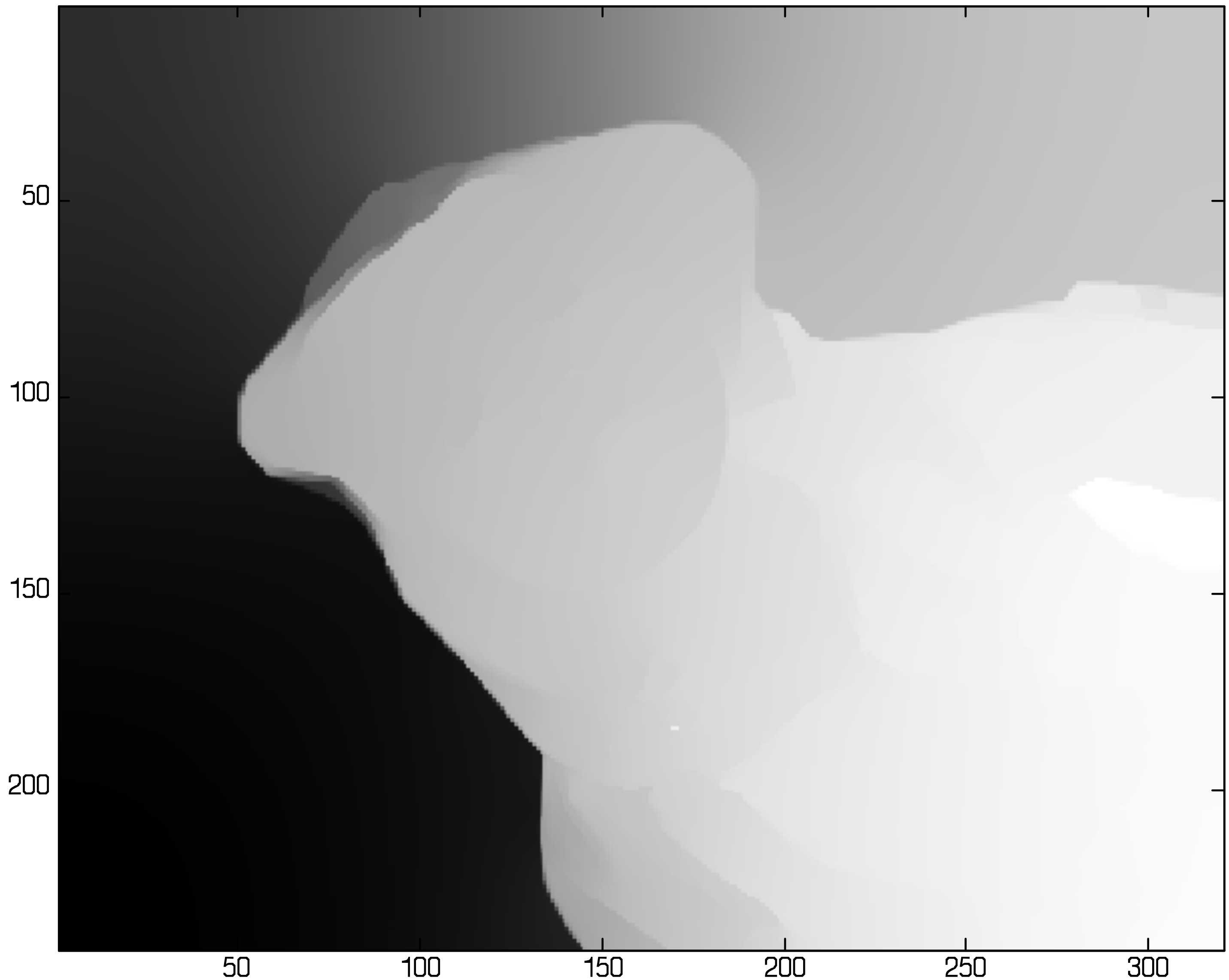


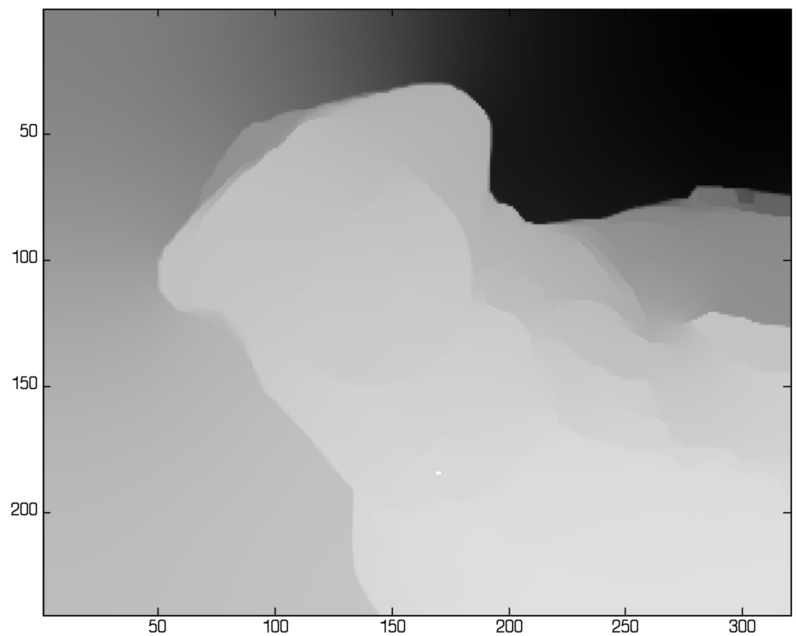
edge weight $e^{-\mathrm{diff}(\mathrm{pixel}\,i,\mathrm{pixel}\,j)^2/t^2}$

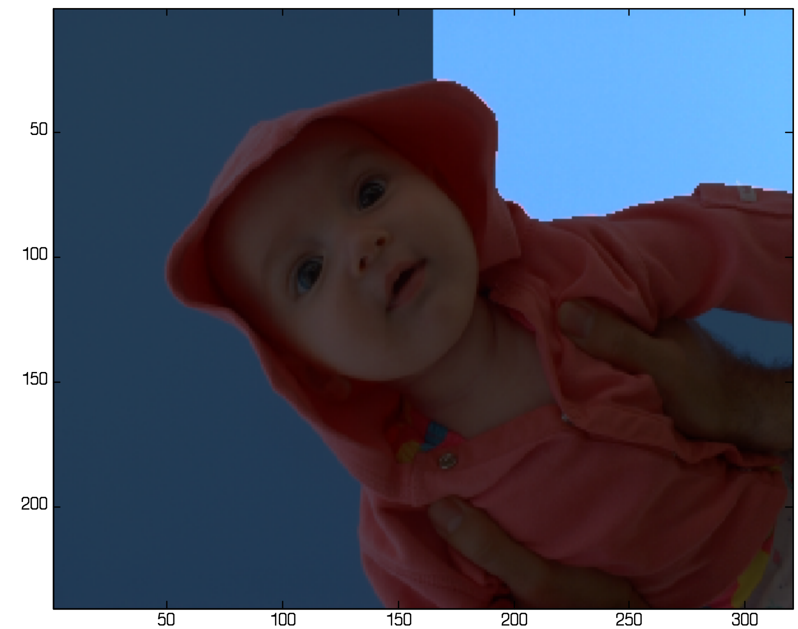
The second eigenvector



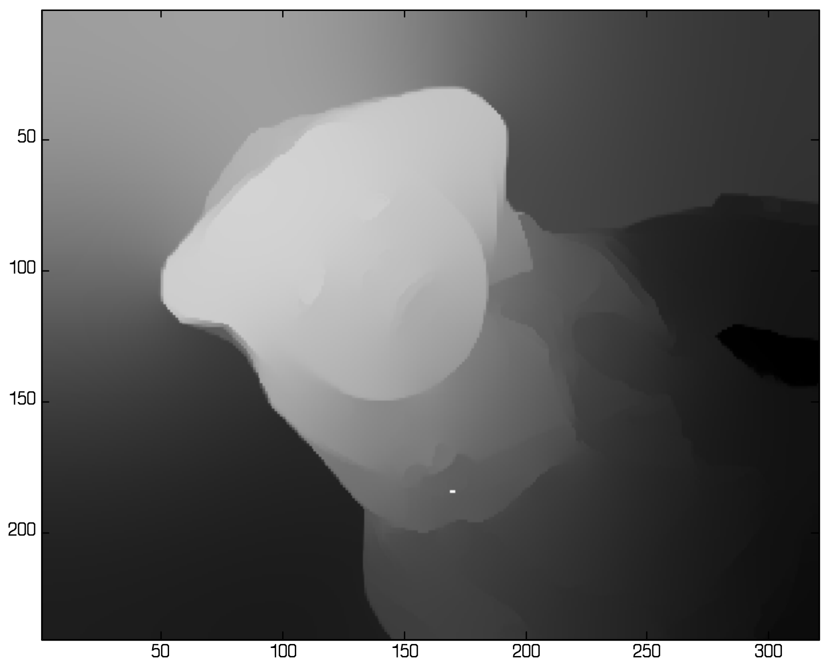
Second eigenvector cut

Third Eigenvector





Fourth Eigenvector



Cheeger's Inequality [Cheeger '70]

[Alon-Milman '85, Jerrum-Sinclair '89, Diaconis-Stroock '91]

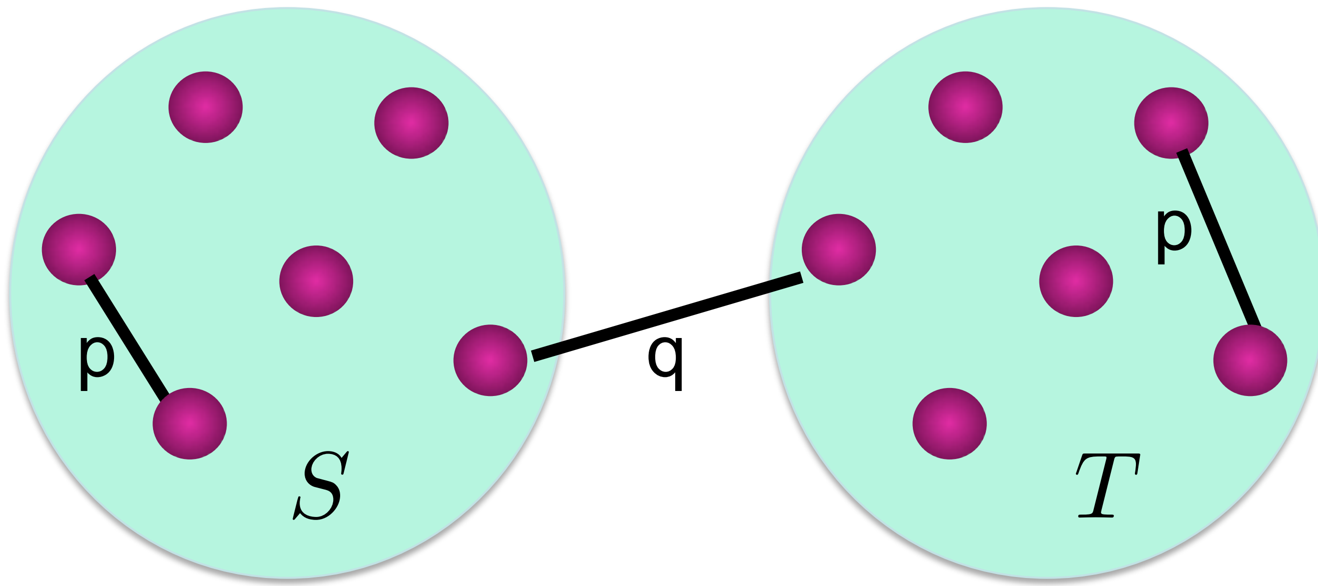
For Normalized Laplacian: $\mathcal{L} = D^{-1/2} L D^{-1/2}$

$$\lambda_2 / 2 \leq \min_S \Phi(S) \leq \sqrt{2\lambda_2}$$

And, is a spectral cut for which

$$\Phi(S) \leq \sqrt{2\lambda_2}$$

McSherry's Analysis of Spectral Partitioning



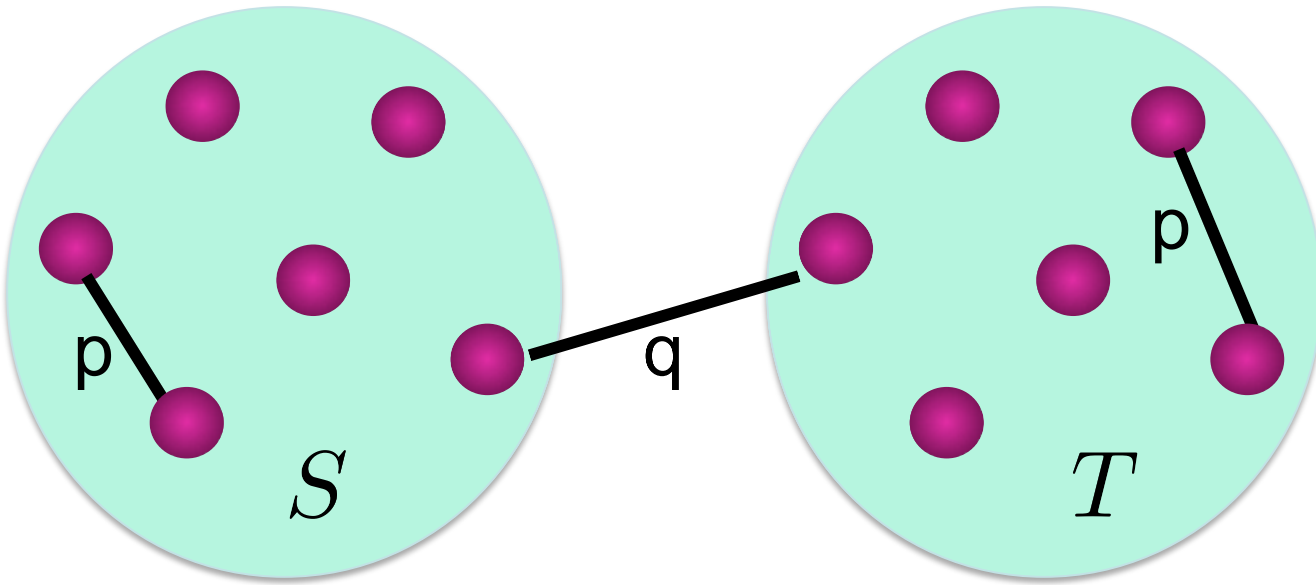
Divide vertices into S and T
Place edges at random with

$$\Pr[\text{S-S edge}] = p$$

$$\Pr[\text{T-T edge}] = p \qquad\qquad q < p$$

$$\Pr[\text{S-T edge}] = q$$

McSherry's Analysis of Spectral Partitioning



$$\mathbf{E}\left[\ A\ \right] = \begin{array}{c} \left(\begin{array}{c|c} p & q \\ \hline q & p \end{array}\right) \\ \begin{array}{cc} S & \quad T \end{array} \end{array} \begin{array}{l} \}\, S \\ \}\, T \\ \ \end{array}$$

McSherry's Analysis of Spectral Partitioning

$$\mathbf{E}[\,A\,] = \begin{array}{cc} \left(\begin{array}{c|c} p & q \\ \hline q & p \end{array}\right) & \begin{array}{l} \}\, S \\ \}\, T \end{array} \\ \begin{array}{cc} S & \quad T \end{array} & \end{array}$$

$v_2(\mathbf{E}[\,L\,])$ is positive const on S, negative const on T

View A as perturbation of $\mathbf{E}[\,A\,]$
and L as perturbation of $\mathbf{E}[\,L\,]$

McSherry's Analysis of Spectral Partitioning

$v_2(\mathbf{E}[\ L\ ])$ is negative const on S, positive const on T

View A as perturbation of $\mathbf{E}[\ A\ ]$
and L as perturbation of $\mathbf{E}[\ L\ ]$

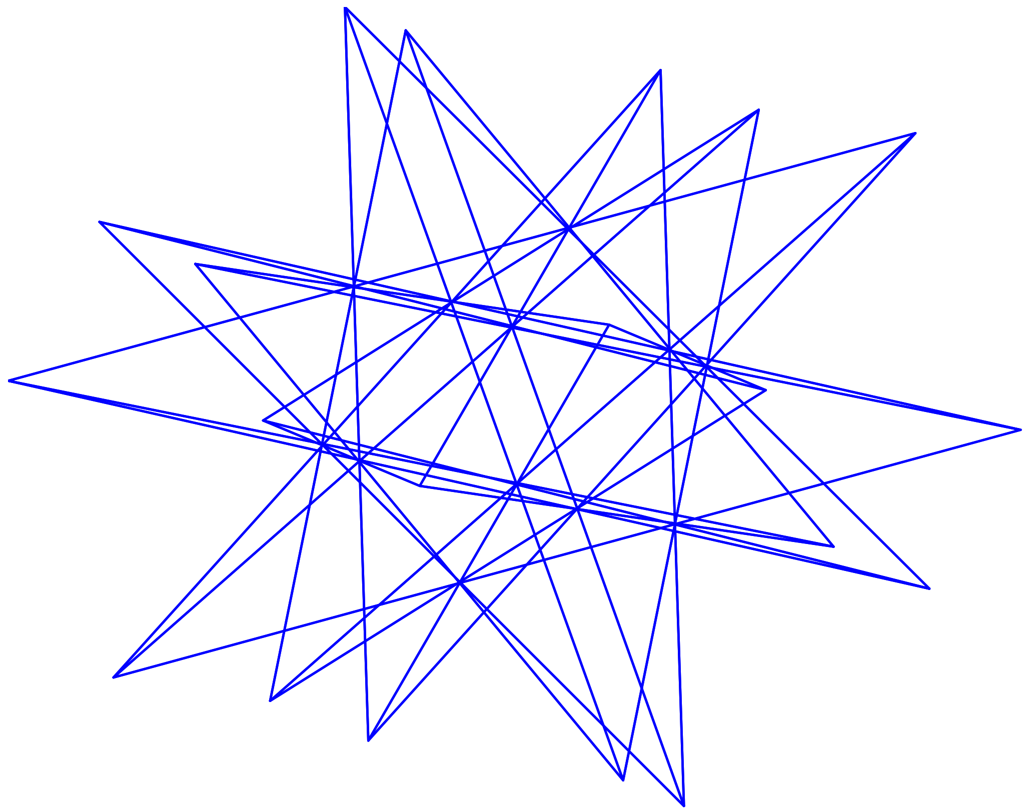
Random Matrix Theory [Füredi-Komlós '81, Vu '07]

With high probability $\left\| L - \mathbf{E}[\ L\ ] \right\|$ small

Perturbation Theory for Eigenvectors implies

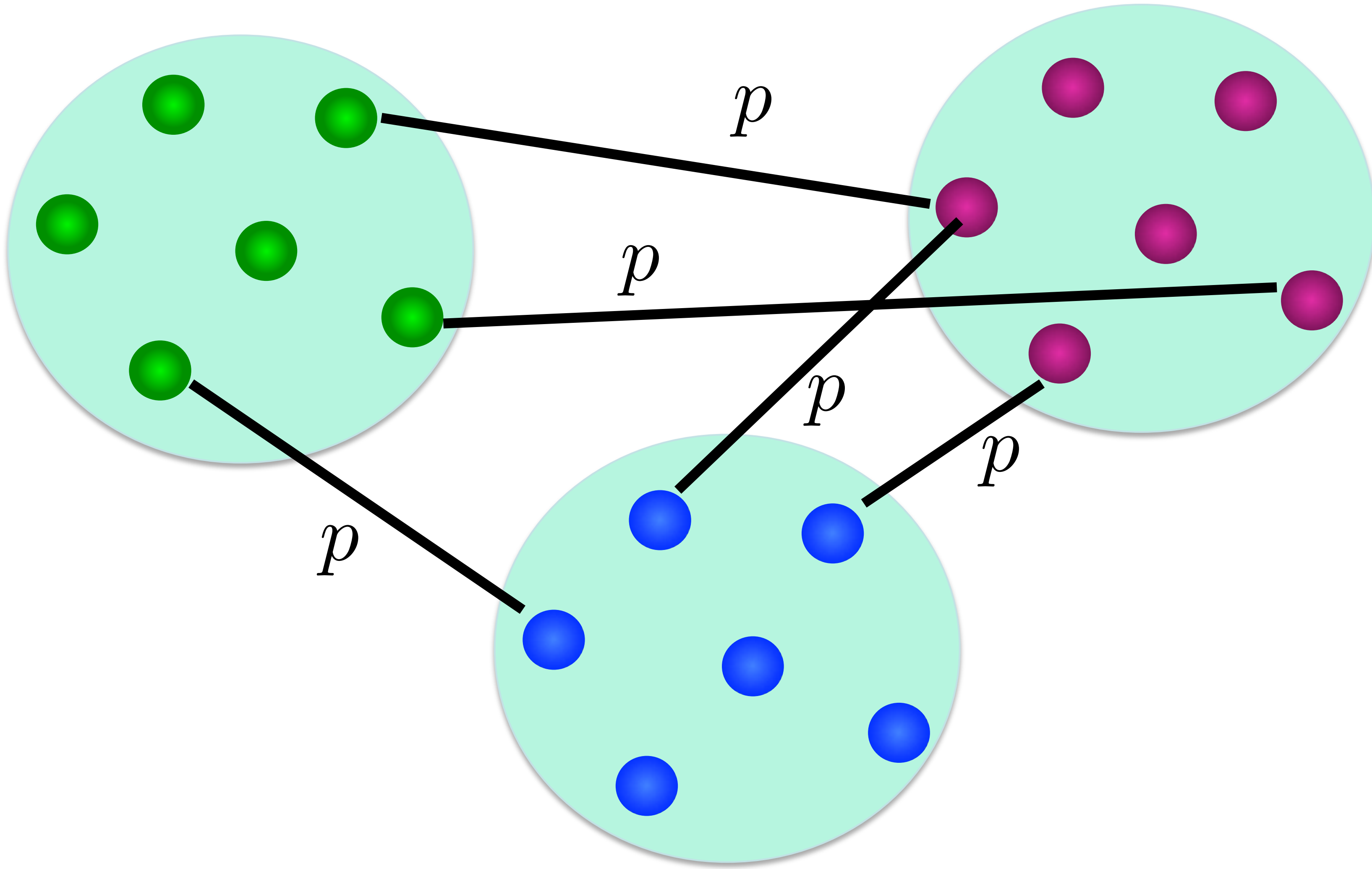
$$v_2(L) \approx v_2(\mathbf{E}[\ L\ ])$$

Spectral graph coloring from high eigenvectors



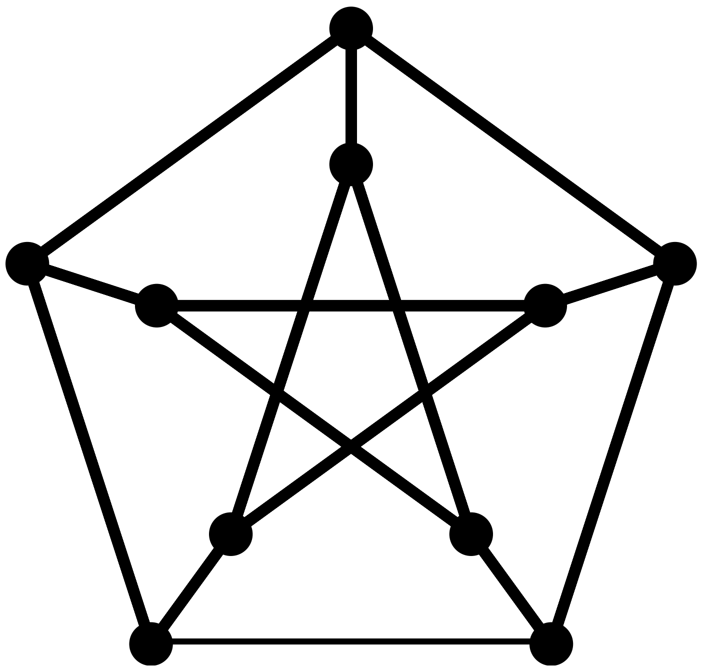
Embedding of dodecahedron by 19th and 20th eigvecs.

Spectral graph coloring from high eigenvectors



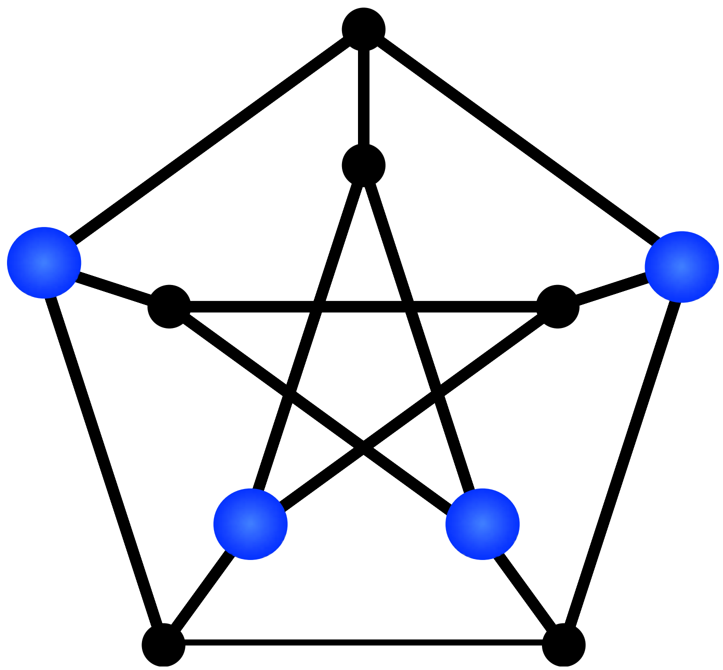
Coloring 3-colorable random graphs [Alon-Kahale '97]

Independent Sets



S is independent if
are no edges between
vertices in S

Independent Sets



S is independent if are no edges between vertices in S

Hoffman's Bound: if every vertex has degree d

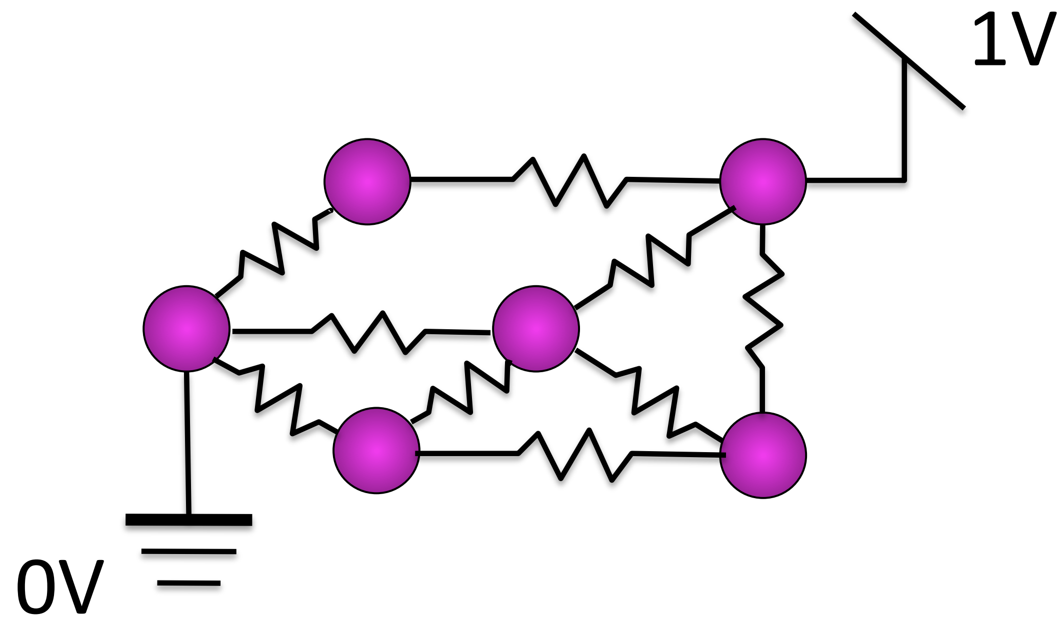
$$|S| \leq n\left(1 - \frac{d}{\lambda_n}\right)$$

Networks of Resistors

Ohm's laws gives $i = v/r$

In general, $\boldsymbol{i} = L_G \boldsymbol{v}$ with $w_{(u,v)} = 1/r_{(u,v)}$

Minimize dissipated energy $\boldsymbol{v}^T L_G \boldsymbol{v}$



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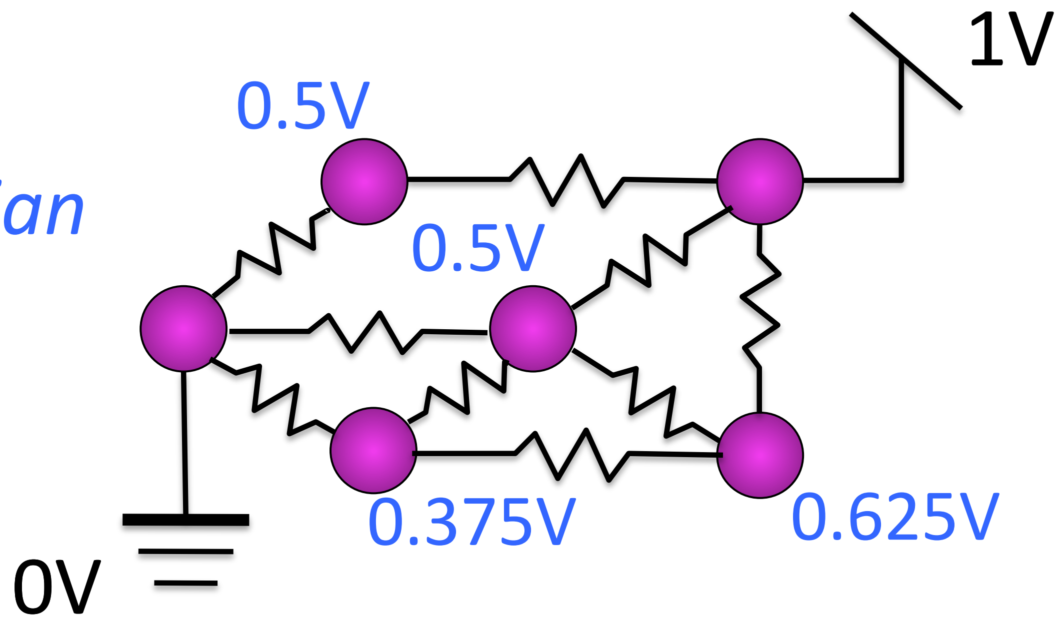
Minimize dissipated energy $\boldsymbol{v}^T L_G \boldsymbol{v}$

By solving Laplacian



Electrical Graph Theory

Considers flows in graphs

Allows comparisons of graphs,
and embedding of one graph within another.

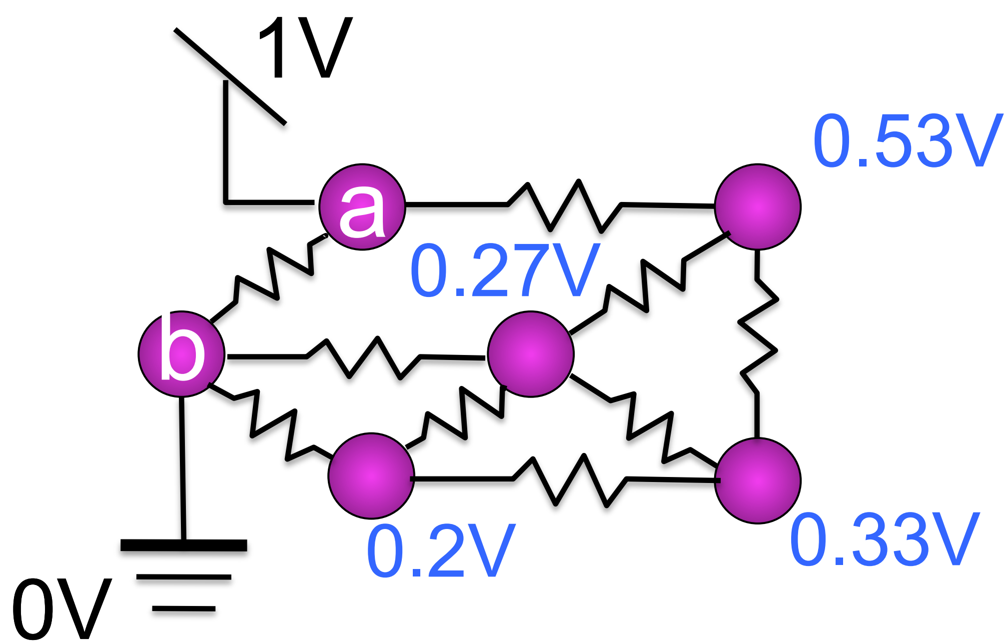
Relative Spectral Graph Theory

Effective Resistance

Resistance of entire network, measured between a and b.

Ohm's law: $r = v/i$

$R_{\mathrm{eff}}(a, b)$ = 1/(current flow at one volt)



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Effective Resistance

$R_{\text{eff}}(a, b)$ = voltage difference to flow 1 unit

Vector of one unit flow has 1 at a,
-1 at b,
0 elsewhere

$$i_{a,b} = e_a - e_b$$

Voltages required by this flow are given by

$$v_{a,b} = L_G^{-1} i_{a,b}$$

Effective Resistance

$R_{\mathrm{eff}}(a,b)$ = voltage difference of unit flow

Voltages required by unit flow are given by

$$v_{a,b} = L_G^{-1} i_{a,b}$$

Voltage difference is

$$\begin{aligned} v_{a,b}(a) - v_{a,b}(b) &= (e_a - e_b)^T v_{a,b} \\ &= (e_a - e_b)^T L_G^{+} (e_a - e_b) \end{aligned}$$

Effective Resistance Distance

Effective resistance is a distance
Lower when are more short paths

Equivalent to commute time distance:
expected time for a random walk from a to reach b and then return to a.

See Doyle and Snell,
Random Walks and Electrical Networks

Relative Spectral Graph Theory

For two connected graphs G and H with the same vertex set, consider

$$L_G L_H^{-1}$$

work orthogonal to nullspace
or use pseudoinverse

Allows one to compare G and H

Relative Spectral Graph Theory

For two connected graphs G and H, consider

$$L_G L_H^{-1} = I_{n-1}$$

if and only if $G = H$

Relative Spectral Graph Theory

For two connected graphs G and H, consider

$$L_G L_H^{-1} \approx I_{n-1}$$

if and only if $G \approx H$

Relative Spectral Graph Theory

For two connected graphs G and H, consider

$$\frac{1}{1+\epsilon} \leq \mathrm{eigs}(L_G L_H^{-1}) \leq 1+\epsilon$$

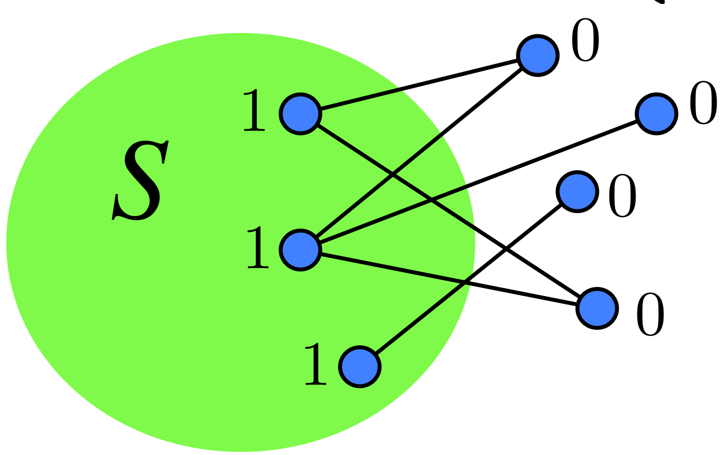
if and only if for all $x \in \mathbb{R}^V$

$$\frac{1}{1+\epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1+\epsilon$$

Relative Spectral Graph Theory

$$\frac{1}{1+\epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1+\epsilon$$

In particular, for $x(a) = \begin{cases} 1 & a \in S \\ 0 & a \notin S \end{cases}$



$$x^T L_G x = \sum_{(a,b) \in E} (x(a) - x(b))^2 = |E(S, V-S)|$$

Relative Spectral Graph Theory

$$\frac{1}{1+\epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1+\epsilon$$

For all $S \subset V$

$$\frac{1}{1+\epsilon} \leq \frac{|E_G(S, V-S)|}{|E_H(S, V-S)|} \leq 1+\epsilon$$

Expanders Approximate Complete Graphs

Expanders:

d-regular graphs on n vertices

high conductance

random walks mix quickly

weak expanders: eigenvalues bounded from 0

strong expanders: all eigenvalues near d

Expanders Approximate Complete Graphs

For G the complete graph on n vertices.

all non-zero eigenvalues of L_G are n.

For $x \perp \mathbf{1}$, $\|x\| = 1$ $\quad x^T L_G x = n$

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For G the complete graph on n vertices.
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For H a d-regular strong expander,
all non-zero eigenvalues of L_H are close to d.

For $x \perp \mathbf{1}$, $\|x\| = 1$ $\qquad x^T L_H x \in [\lambda_2, \lambda_n]$

$$\approx d$$

Expanders Approximate Complete Graphs

For G the complete graph on n vertices.
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For $x \perp \mathbf{1}$, $\|x\| = 1$ $\qquad x^T L_G x = n$

For H a d-regular strong expander,
all non-zero eigenvalues of L_H are close to d.

For $x \perp \mathbf{1}$, $\|x\| = 1$ $\qquad x^T L_H x \approx d$

$\frac{n}{d} H$ is a good approximation of G

Sparse approximations of every graph

$$\frac{1}{1+\epsilon} \leq \frac{x^T L_G x}{x^T L_H x} \leq 1+\epsilon$$

For every G,
there is an H with $(2+\epsilon)^2 n/\epsilon^2$ edges
[Batson-S-Srivastava]

Can find an H with $O(n \log n/\epsilon^2)$ edges
in nearly-linear time.
[S-Srivastava]

Sparsification by Random Sampling [S-Srivastava]

Include edge (u, v) with probability

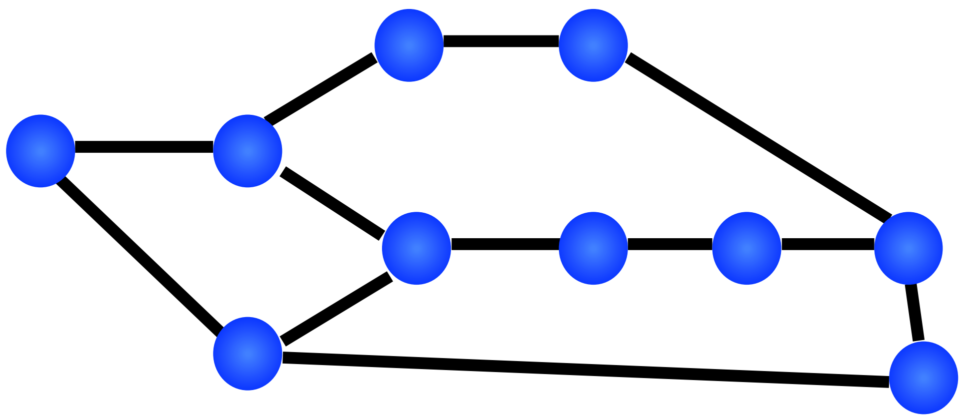
$$p_{u,v} \sim w_{u,v} R_{\mathrm{eff}}(u, v)$$

If include edge, give weight $w_{u,v}/p_{u,v}$

Analyze by Rudelson's
concentration of random sums of rank-1 matrices

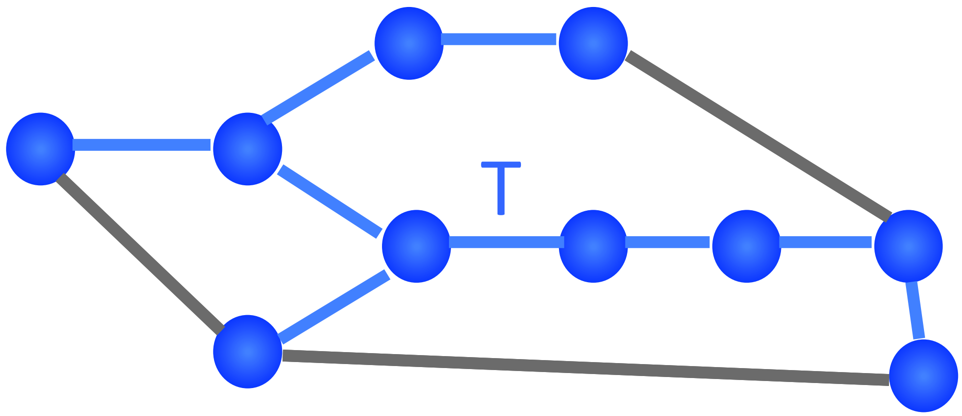
Approximating a graph by a tree

Alon, Karp, Peleg, West ‘91: measure the stretch



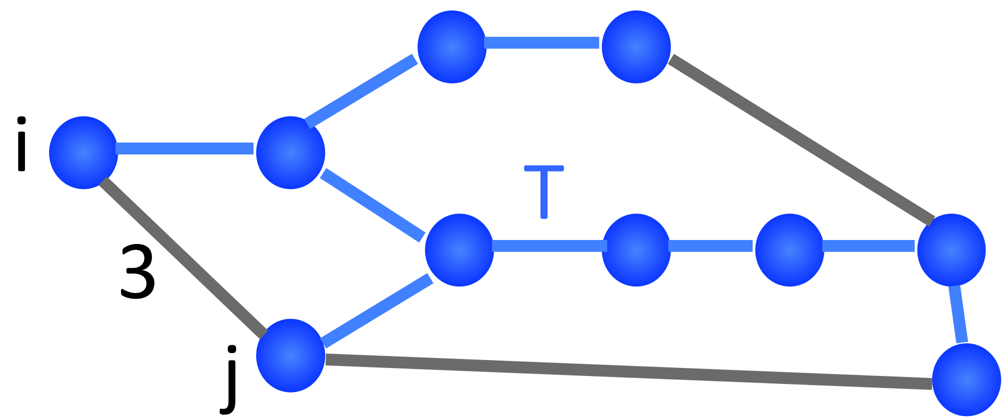
Approximating a graph by a tree

Alon, Karp, Peleg, West ‘91: measure the stretch



Approximating a graph by a tree

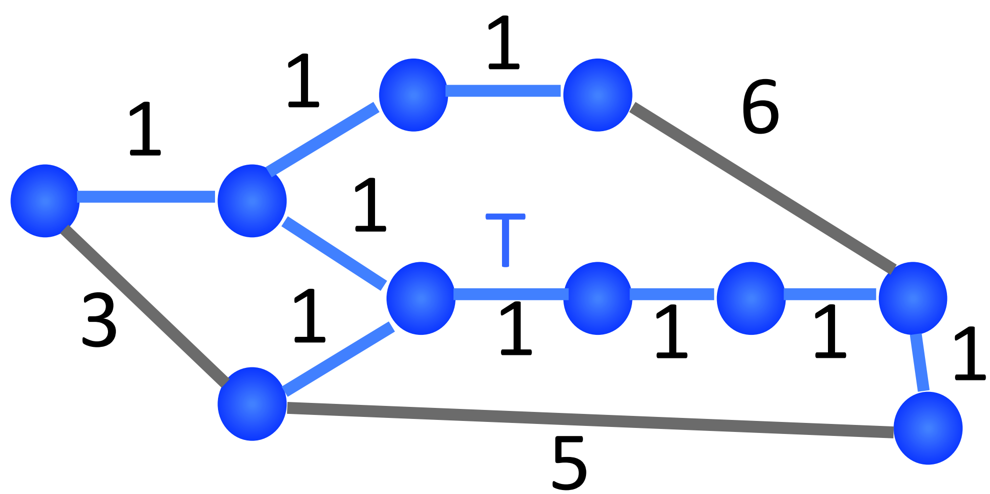
Alon, Karp, Peleg, West '91: measure the stretch



$$\mathrm{stretch}_T(i,j) = \mathrm{dist}_T(i,j)$$

Approximating a graph by a tree

Alon, Karp, Peleg, West '91: measure the stretch



$$\mathrm{stretch}_T(G) = \sum_{(i,j)\in G} \mathrm{dist}_T(i,j)$$

Low-Stretch Spanning Trees

For every G there is a T with

$$\mathrm{stretch}_T(G) \leq m^{1+o(1)}$$ where $m = |E|$

(Alon-Karp-Peleg-West '91)

$$\mathrm{stretch}_T(G) \leq O(m \log m \log^2 \log m)$$

(Elkin-Emek-S-Teng '04, Abraham-Bartal-Neiman '08)

Conjecture: $\mathrm{stretch}_T(G) \leq m \log_2 m$

Algebraic characterization of stretch [S-Woo '09]

$$\mathrm{stretch}_T(G) = \mathrm{Trace}[L_G L_T^{-1}]$$

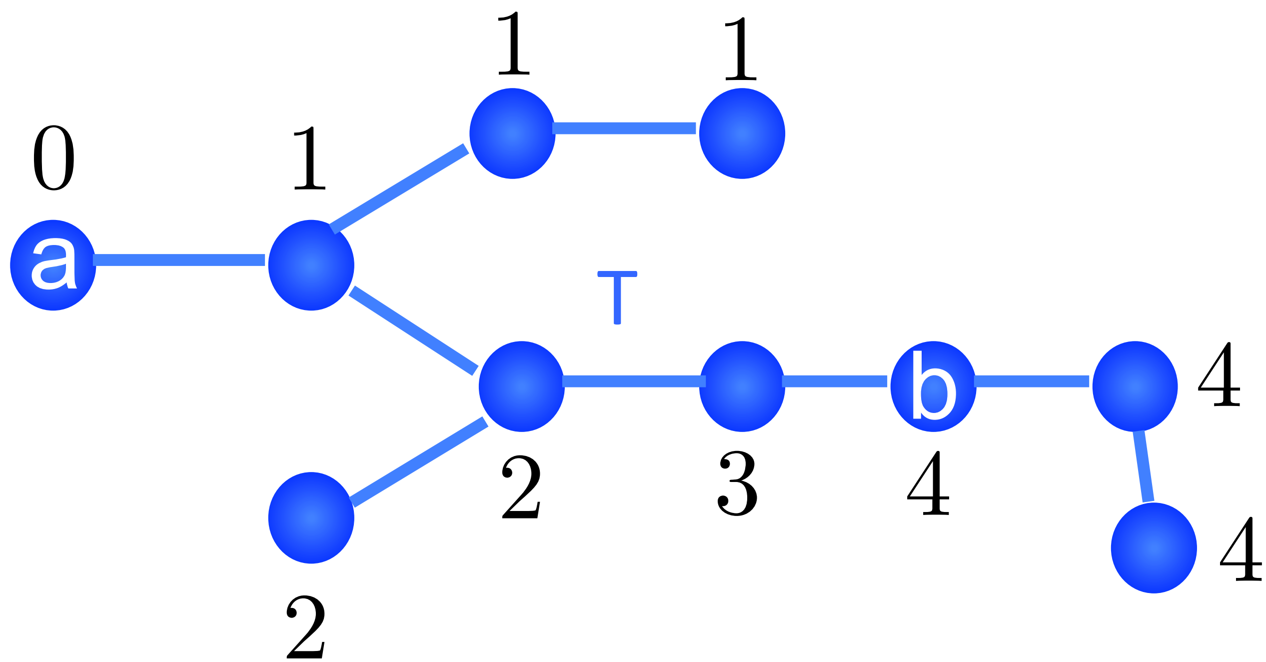
Algebraic characterization of stretch [S-Woo '09]

$$\mathrm{stretch}_T(G) = \mathrm{Trace}[L_G L_T^{-1}]$$

Resistances in series sum

In trees, resistance is distance.

v :



Algebraic characterization of stretch [S-Woo '09]

$$\mathrm{stretch}_T(G) = \mathrm{Trace}[L_G L_T^{-1}]$$

$$\begin{aligned}
x^T L_G x &= \sum_{(a,b)\in E} (x(a) - x(b))^2 \\
&= \sum_{(a,b)\in E} ((e_a - e_b)^T x)^2 \\
&= \sum_{(a,b)\in E} x^T (e_a - e_b)(e_a - e_b)^T x \\
&= x^T (\sum_{(a,b)\in E} (e_a - e_b)(e_a - e_b)^T) x
\end{aligned}$$

Algebraic characterization of stretch [S-Woo '09]

$$\mathrm{stretch}_T(G) = \mathrm{Trace}[L_G L_T^{-1}]$$

$$\begin{aligned}
\mathrm{Trace}[L_G L_T^{-1}] &= \sum_{(a,b)\in E} \mathrm{Trace}[(e_a - e_b)(e_a - e_b)^T L_T^{-1}] \\
&= \sum_{(a,b)\in E} \mathrm{Trace}[(e_a - e_b)^T L_T^{-1}(e_a - e_b)] \\
&= \sum_{(a,b)\in E} (e_a - e_b)^T L_T^{-1}(e_a - e_b)
\end{aligned}$$

Algebraic characterization of stretch [S-Woo '09]

$$\mathrm{stretch}_T(G) = \mathrm{Trace}[L_G L_T^{-1}]$$

$$\sum_{(a,b)\in E} (e_a - e_b)^T L_T^{-1} (e_a - e_b) = \sum_{(a,b)\in E} R_{\mathrm{eff}}(a,b)$$

$$= \sum_{(a,b)\in E} \mathrm{stretch}_T(a,b)$$

Notable Things I've left out

Behavior under graph transformations
Graph Isomorphism
Random Walks and Diffusion
PageRank and Hits
Matrix-Tree Theorem
Special Graphs
(Cayley, Strongly-Regular, etc.)
Diameter bounds
Colin de Verdière invariant
Discretizations of Manifolds

The next two talks

Tomorrow:
Solving equations in Laplacians
in nearly-linear time.

Preconditioning
Sparsification
Low-Stretch Spanning Trees
Local graph partitioning

The next two talks

Thursday:
Existence of sparse approximations.

A theorem in linear algebra
and some of its connections.