Ramanujan Graphs of Every Degree

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Expander Graphs

Sparse, regular well-connected graphs
with many properties of random graphs.

Random walks mix quickly.

Every set of vertices has many neighbors.

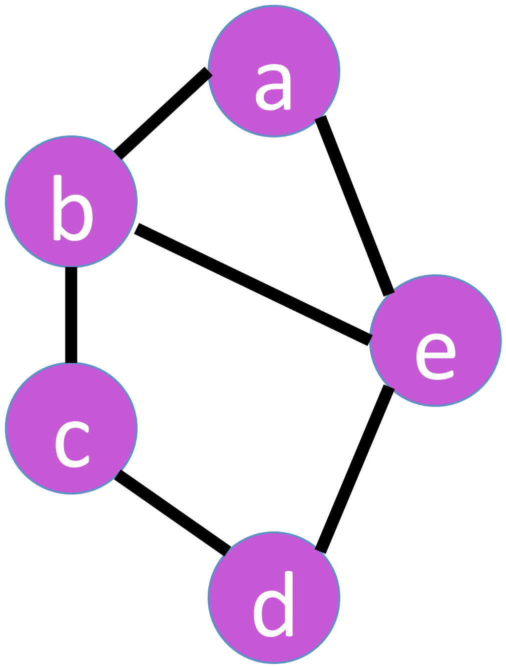
Pseudo-random generators.

Error-correcting codes.

Sparse approximations of complete graphs.

Spectral Expanders

Let G be a graph and A be its adjacency matrix



0	1	0	0	1
1	0	1	0	1
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Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

"trivial"

If d-regular (every vertex has d edges), $\lambda_1 = d$

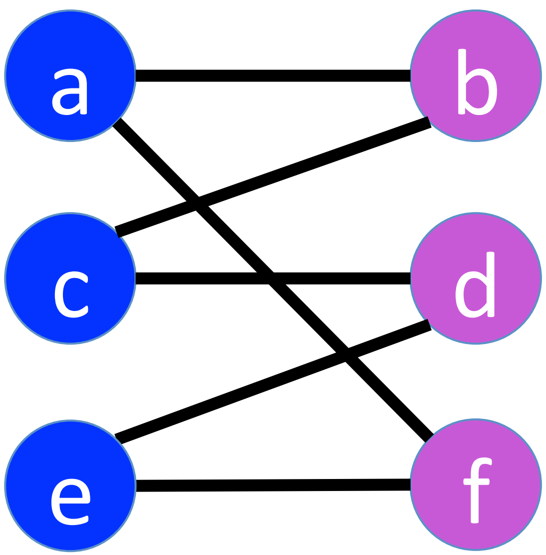
Spectral Expanders

If bipartite (all edges between two parts/colors)

eigenvalues are symmetric about 0

If d-regular and bipartite, $\lambda_n = -d$

"trivial"



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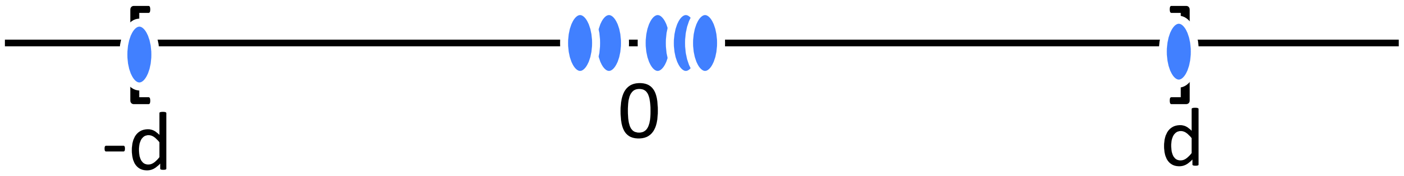
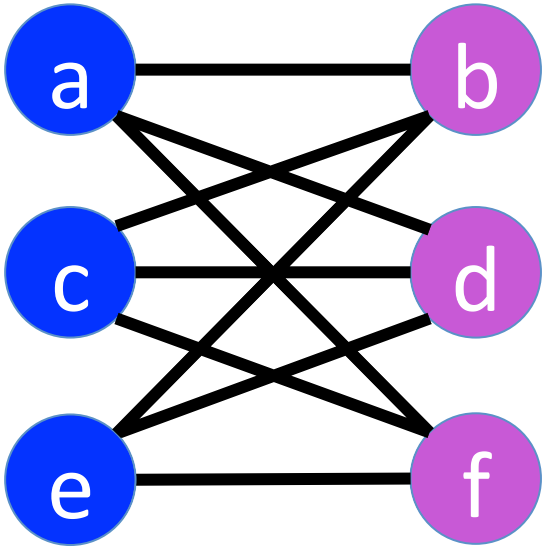
Spectral Expanders

G is a good spectral expander
if all non-trivial eigenvalues are small



Bipartite Complete Graph

Adjacency matrix has rank 2,
so all non-trivial eigenvalues are 0



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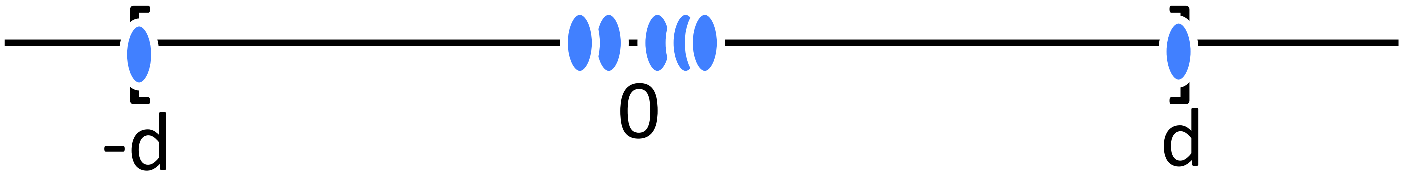

Challenge:
construct infinite families of fixed degree

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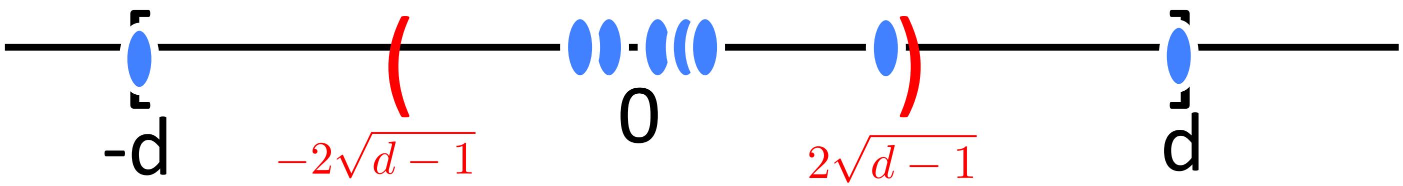
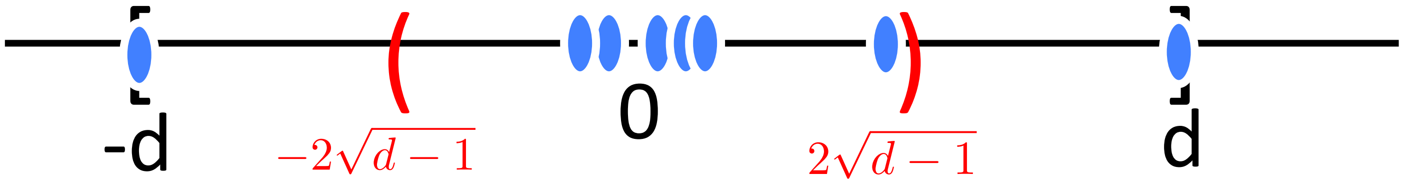


Challenge:
construct infinite families of fixed degree

Alon-Boppana '86: Cannot beat $2\sqrt{d-1}$

Ramanujan Graphs: $2\sqrt{d-1}$

G is a Ramanujan Graph
if absolute value of non-trivial eigs $\leq 2\sqrt{d-1}$



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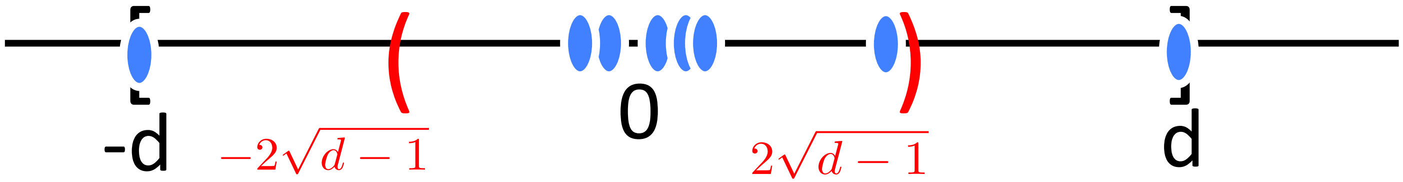

Margulis, Lubotzky-Phillips-Sarnak'88: Infinite sequences of Ramanujan graphs exist for $d = \text{prime} + 1$

Ramanujan Graphs: $2\sqrt{d-1}$

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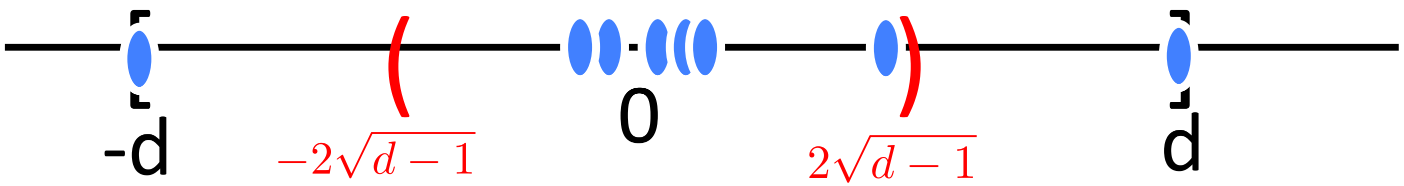

Friedman'08: A random d-regular graph is almost Ramanujan : $2\sqrt{d-1} + \epsilon$

Ramanujan Graphs of Every Degree

Theorem:
there are infinite families of
bipartite Ramanujan graphs of every degree.

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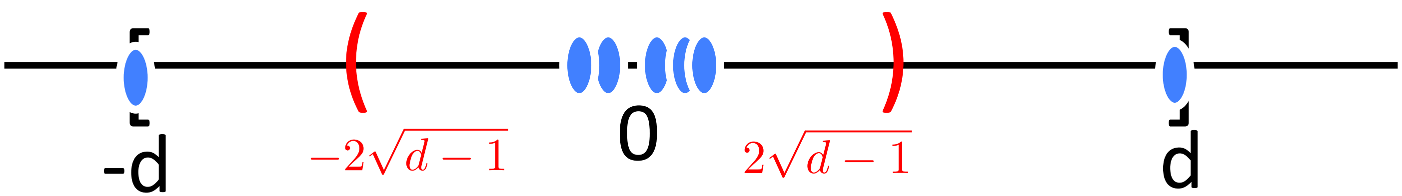
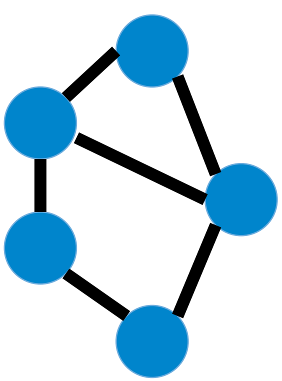
And, are infinite families of (c,d)-biregular
Ramanujan graphs, having non-trivial eigs
bounded by

$$\sqrt{d-1} + \sqrt{c-1}$$

Bilu-Linial '06 Approach

Find an operation that doubles the size of a graph without creating large eigenvalues.





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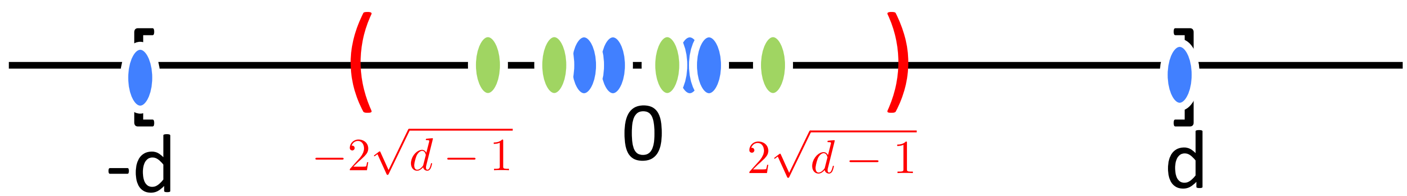
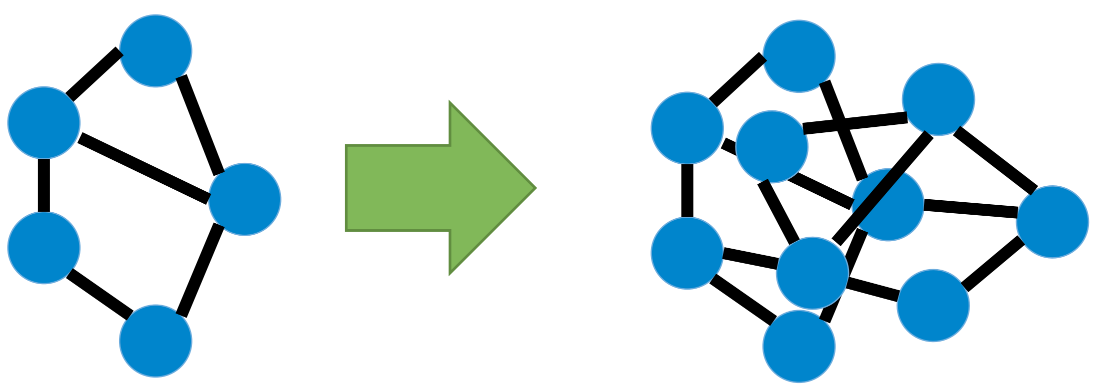
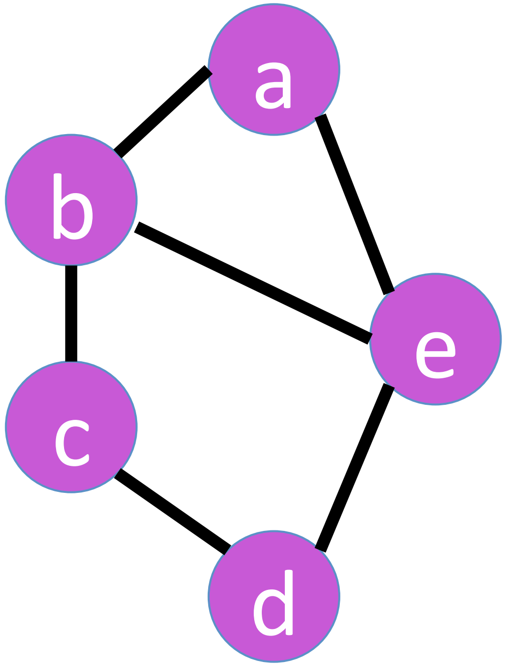




2-lifts of graphs



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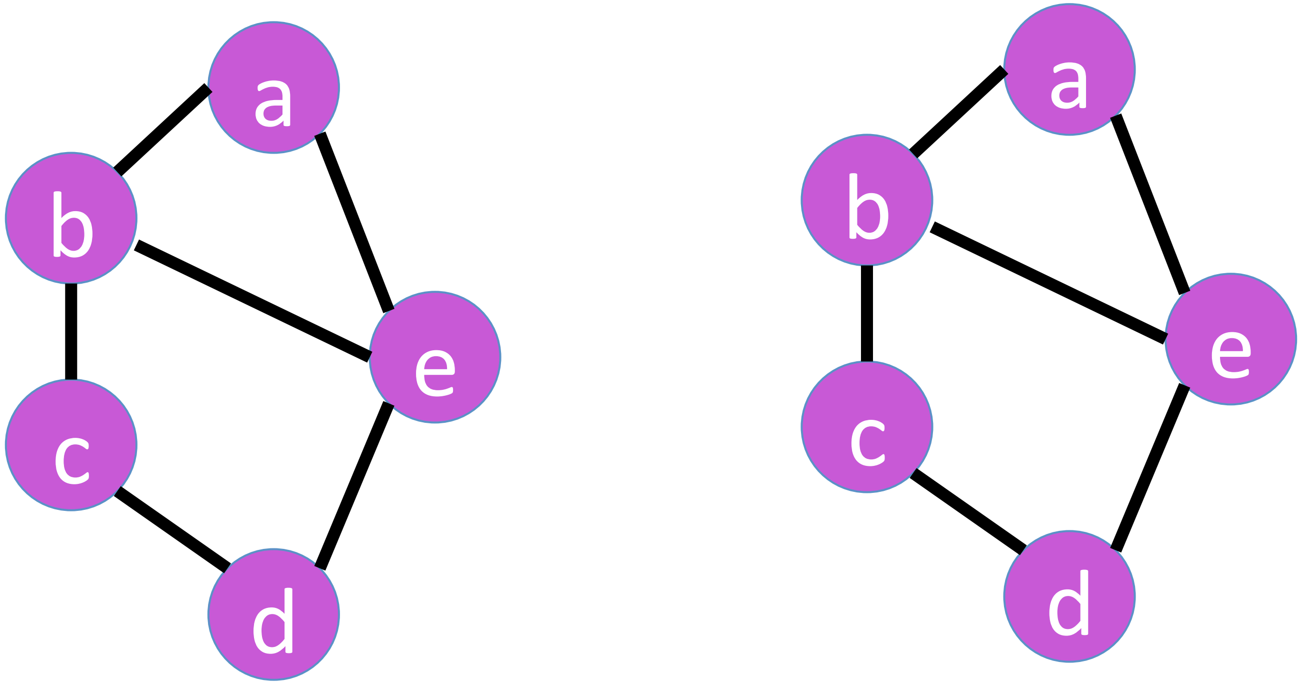

duplicate every vertex

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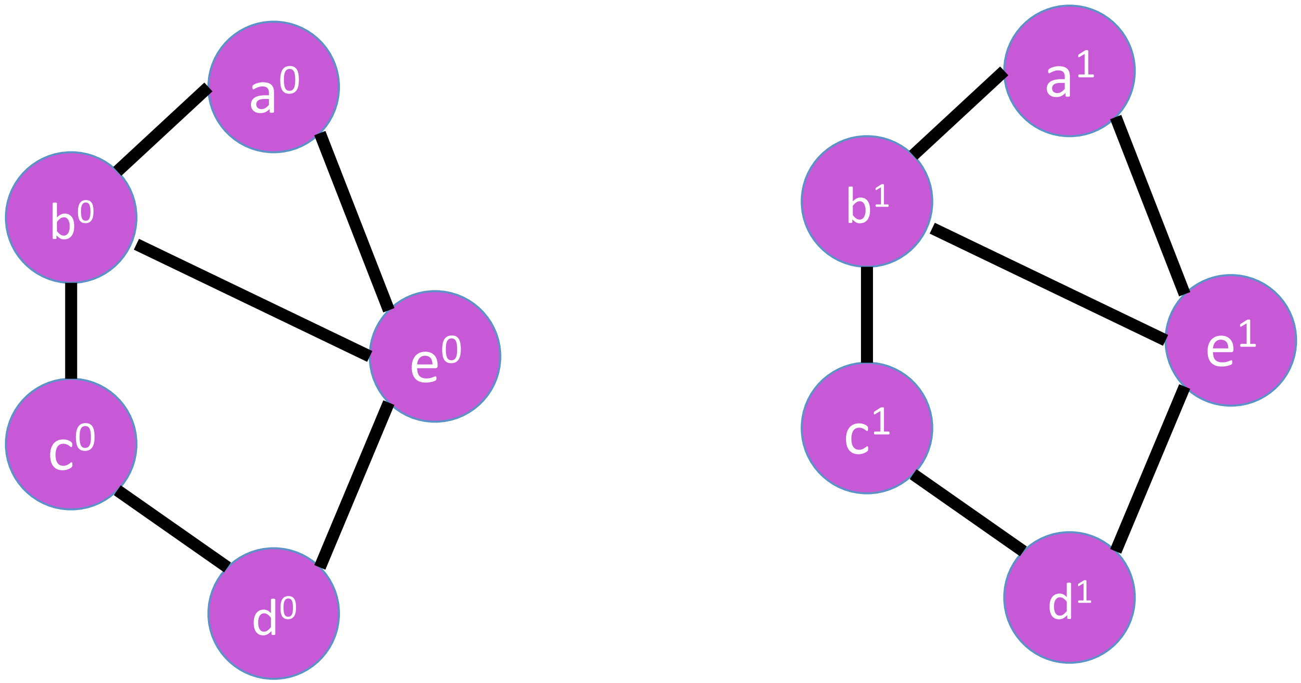

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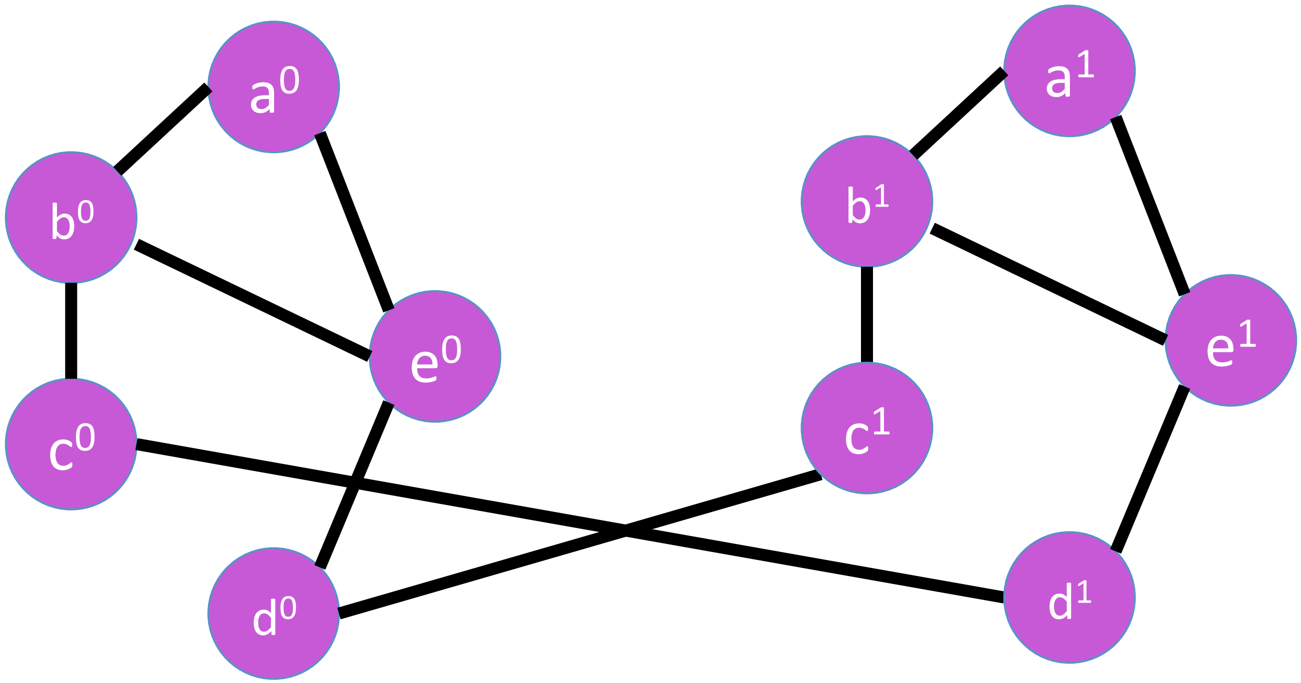

for every pair of edges:
leave on either side (parallel),
or make both cross

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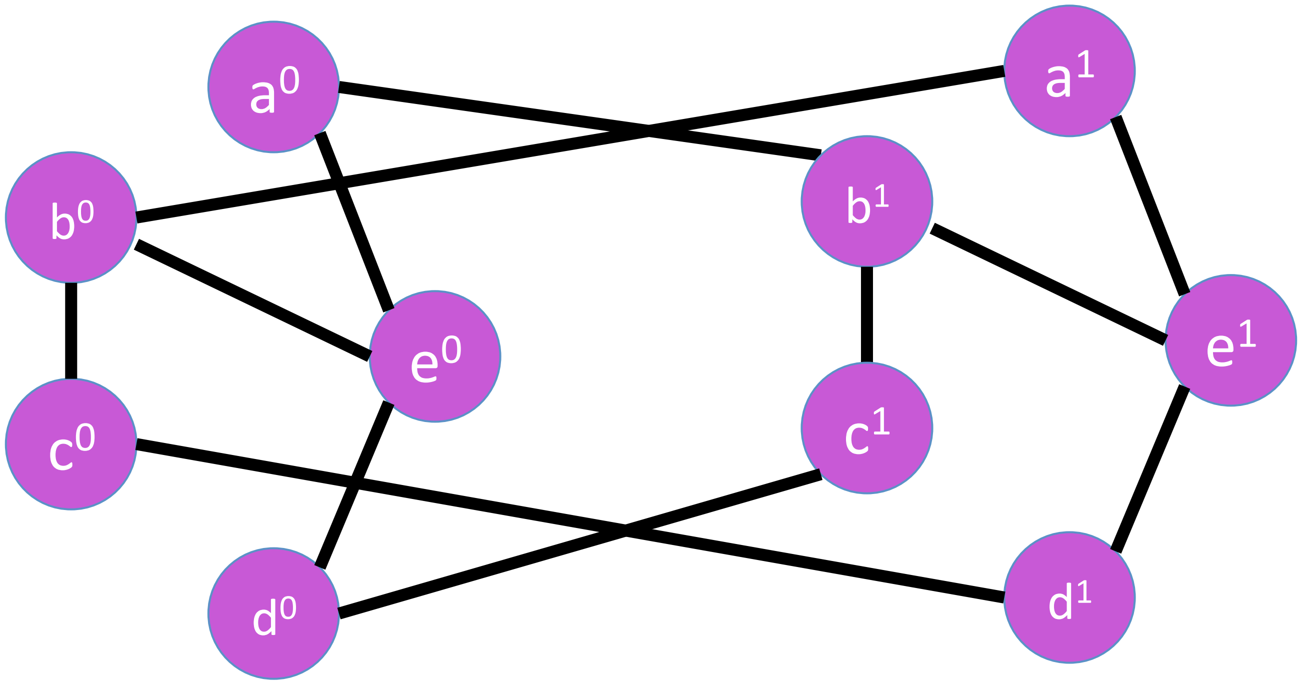

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1	1	0	1	0	0	0	0	0	0
0	0	0	0	0	0	1	0	0	1
0	0	0	0	0	1	0	1	0	1
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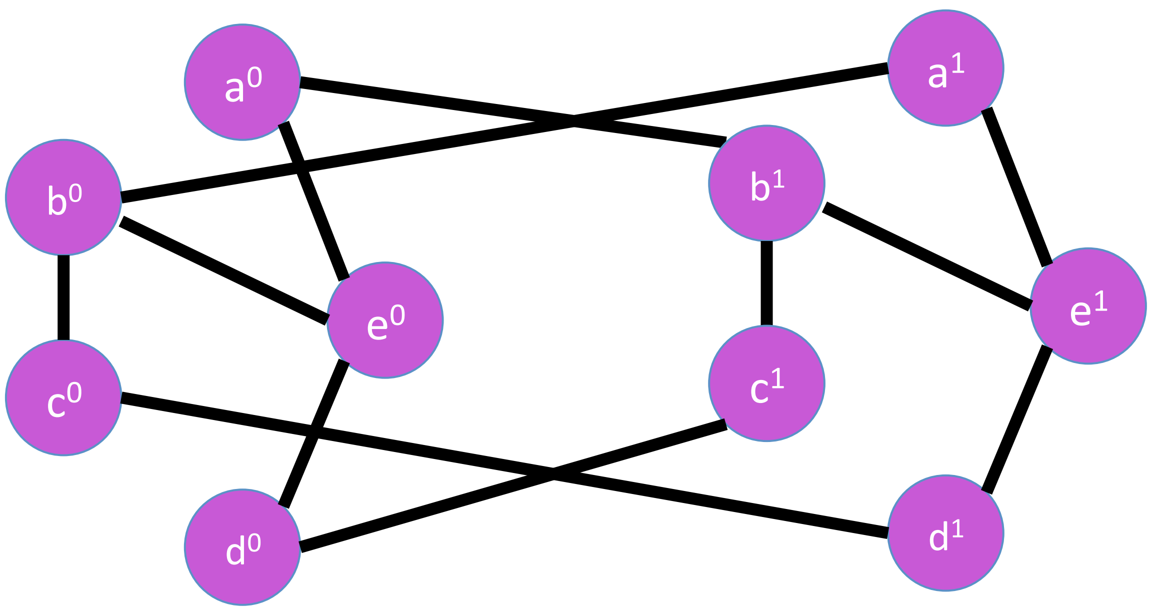
1	1	0	1	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	1
1	0	0	0	0	0	0	1	0	1
0	0	0	1	0	0	1	0	0	0
0	0	1	0	0	0	0	0	0	1
0	0	0	0	0	1	1	0	1	0

Eigenvalues of 2-lifts (Bilu-Linial)

Given a 2-lift of G,
create a signed adjacency matrix A_s
with a -1 for crossing edges
and a 1 for parallel edges



0	-1	0	0	1
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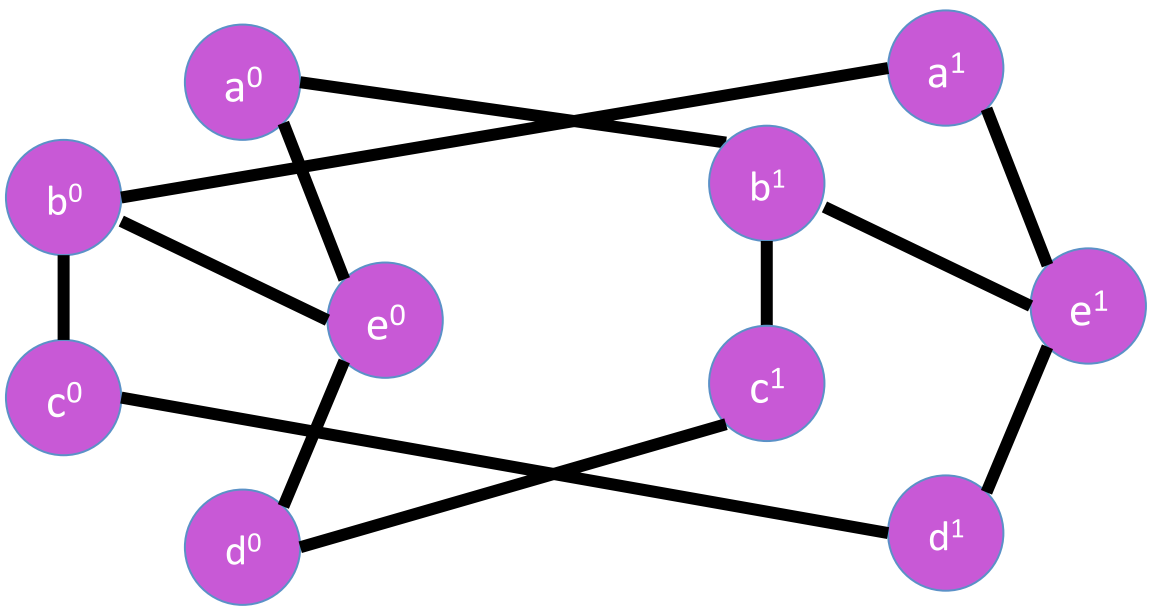
Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:

The eigenvalues of the 2-lift are the union of the eigenvalues of A (old) and the eigenvalues of A_s (new)



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Conjecture:
Every d-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$

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Every d-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$

Would give infinite families of Ramanujan Graphs:

start with the complete graph,
and keep lifting.

Eigenvalues of 2-lifts (Bilu-Linial)

Conjecture:

Every d-regular graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$

We prove this in the bipartite case.

a 2-lift of a bipartite graph is bipartite

Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:

Every d-regular graph has a 2-lift in which all the new eigenvalues have ~~absolute~~ value at most $2\sqrt{d-1}$

Trick: eigenvalues of bipartite graphs are symmetric about 0, so only need to bound largest

Eigenvalues of 2-lifts (Bilu-Linial)

Theorem:

Every d-regular bipartite graph has a 2-lift in which all the new eigenvalues have absolute value at most $2\sqrt{d-1}$

First idea: a random 2-lift

Specify a lift by $s \in \{\pm 1\}^m$

Pick s uniformly at random

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Are graphs for which this usually fails

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Specify a lift by $s \in \{\pm 1\}^m$

Pick s uniformly at random

Are graphs for which this usually fails

Bilu and Linial proved G almost Ramanujan, implies new eigenvalues usually small.

Improved by Puder and Agarwal-Kolla-Madan

The expected polynomial

Consider $\mathop{\mathbb{E}}_{s}\left[\ \chi_{A_s}(x)\ \right]$

The expected polynomial

Consider $\mathop{\mathbb{E}}_{s}[\ \chi_{A_s}(x)\ ]$

Prove max-root $\left(\mathop{\mathbb{E}}_{s}[\ \chi_{A_s}(x)\ ]\right) \leq 2\sqrt{d-1}$

Prove $\chi_{A_s}(x)$ is an interlacing family

Conclude there is an s so that

max-root $(\chi_{A_s}(x)) \leq 2\sqrt{d-1}$

The expected polynomial

Theorem (Godsil-Gutman '81):

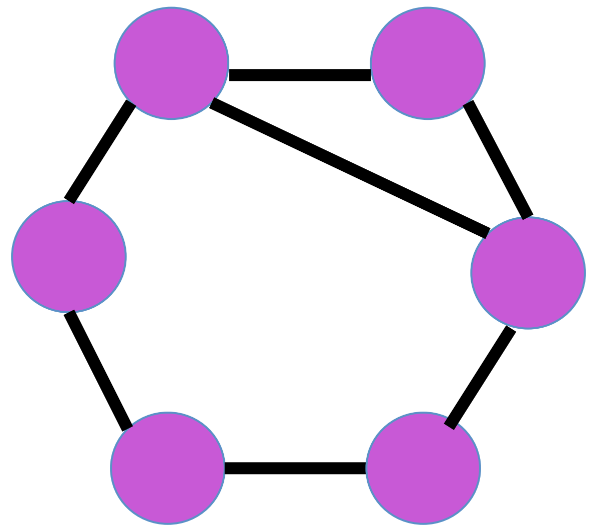
$$\mathop{\mathbb{E}}_{s}\left[\ \chi_{A_s}(x)\ \right] = \mu_G(x)$$

the matching polynomial of G

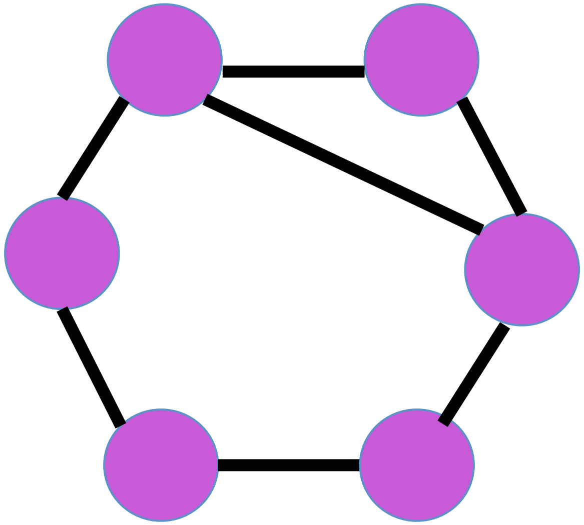
The matching polynomial (Heilmann-Lieb '72)

$$\mu_G(x) = \sum_{i \geq 0} x^{n-2i}(-1)^i m_i$$

m_i = the number of matchings with i edges



$$\mu_G(x) = x^6 - 7x^4 + 11x^2 - 2$$



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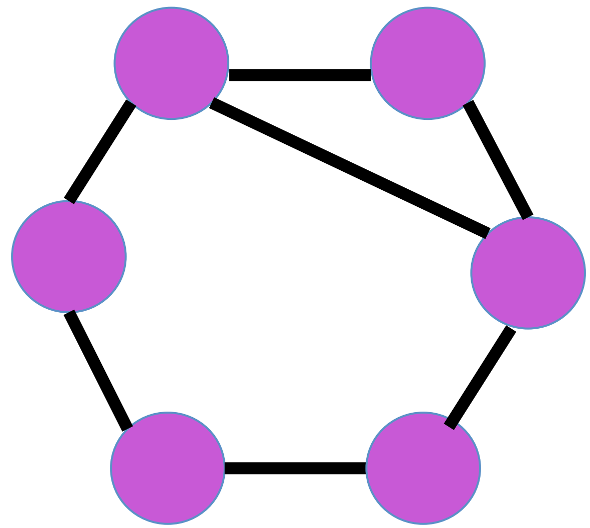
one matching with 0 edges



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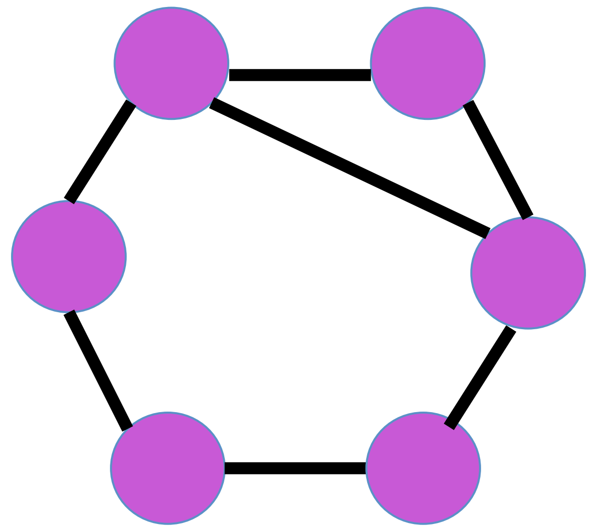
7 matchings with 1 edge



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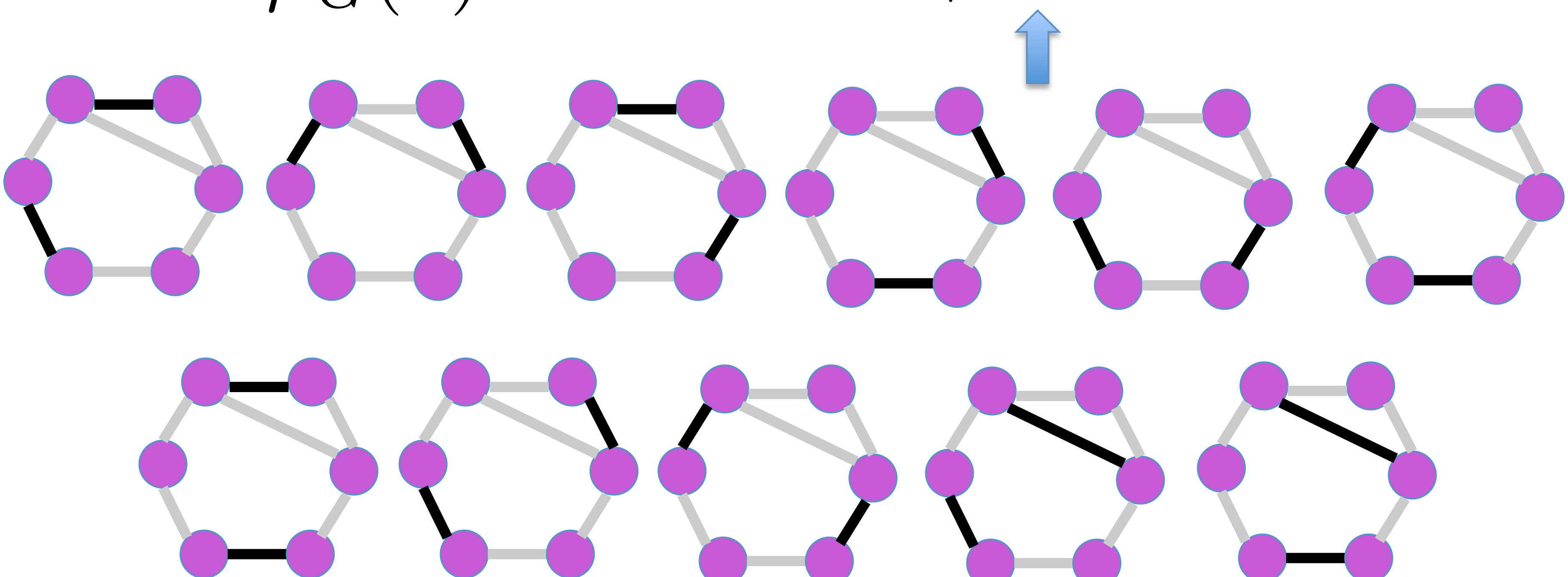

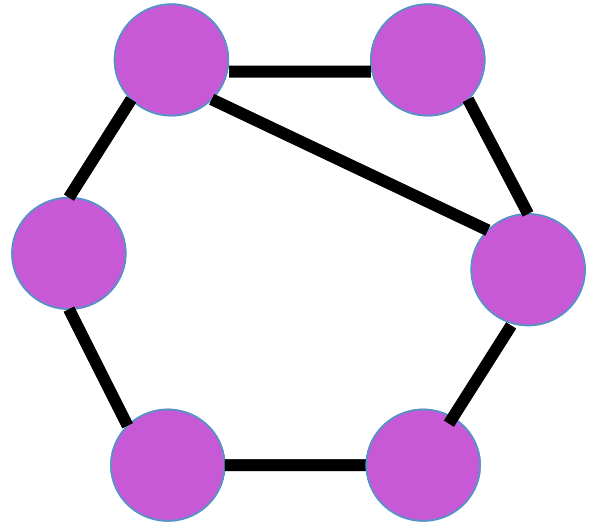

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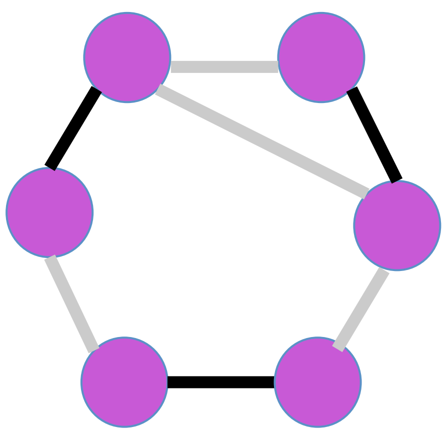

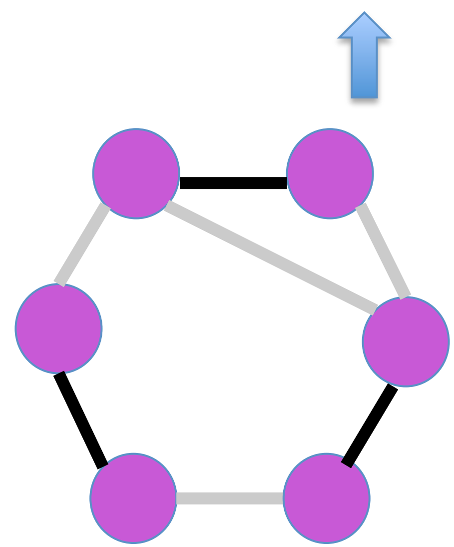

Proof that $\mathop{\mathbb{E}}_{s}[\ \chi_{A_s}(x)\ ] = \mu_G(x)$

Expand $\mathop{\mathbb{E}}_{s}[\ \det(xI - A_s)\ ]$ using permutations

$$
\begin{matrix}
x & \pm 1 & 0 & 0 & \pm 1 & \pm 1 \\
\pm 1 & x & \pm 1 & 0 & 0 & 0 \\
0 & \pm 1 & x & \pm 1 & 0 & 0 \\
0 & 0 & \pm 1 & x & \pm 1 & 0 \\
\pm 1 & 0 & 0 & \pm 1 & x & \pm 1 \\
\pm 1 & 0 & 0 & 0 & \pm 1 & x
\end{matrix}
$$

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same edge: same value

x	±1	0	0	±1	±1
±1	x	±1	0	0	0
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Get 0 if hit any 0s

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Get 0 if take just one entry for any edge

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Only permutations that count are involutions

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Only permutations that count are involutions

Correspond to matchings

The matching polynomial (Heilmann-Lieb '72)

$$\mu_G(x) = \sum_{i \geq 0} x^{n-2i} (-1)^i m_i$$

Theorem (Heilmann-Lieb)
all the roots are real

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Implies max-root $\left(\mathbb{E}_s [\ \chi_{A_s}(x)\ ] \right) \leq 2\sqrt{d-1}$

Interlacing

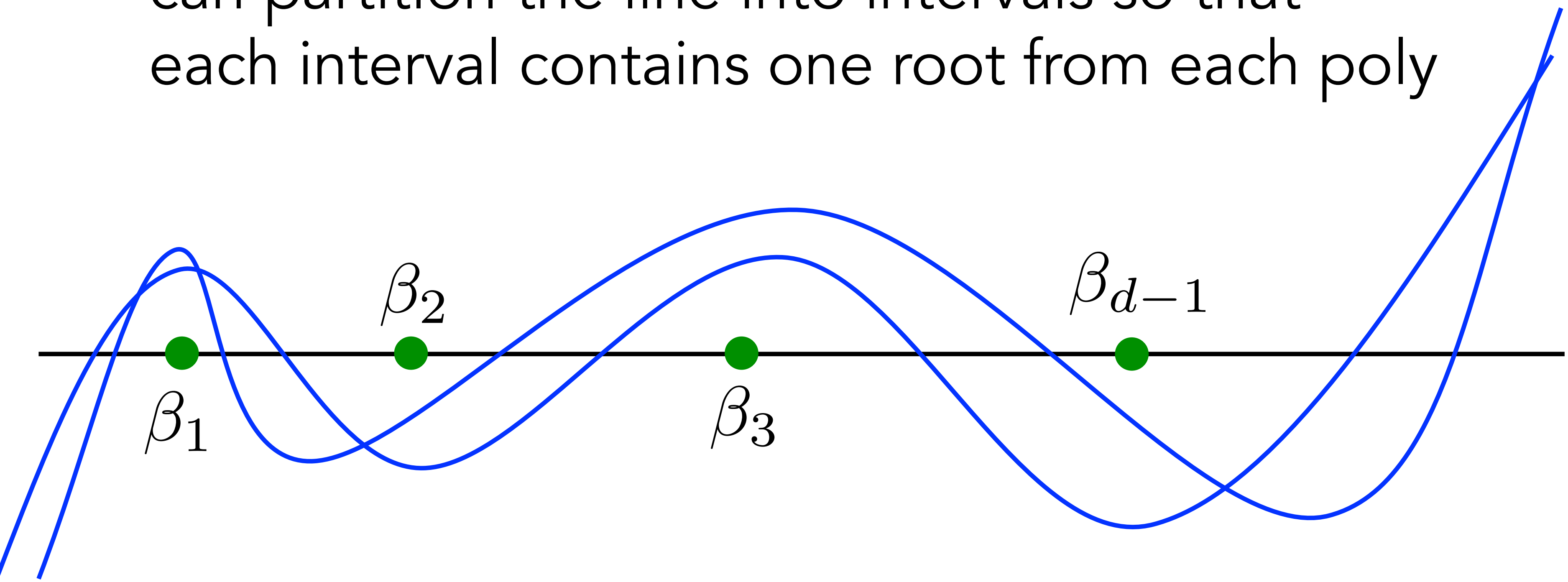
Polynomial $p(x) = \prod_{i=1}^{d} (x - \alpha_i)$

interlaces $q(x) = \prod_{i=1}^{d-1} (x - \beta_i)$

if $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \alpha_{d-1} \leq \beta_{d-1} \leq \alpha_d$

Common Interlacing

$p_1(x)$ and $p_2(x)$ have a common interlacing if can partition the line into intervals so that each interval contains one root from each poly



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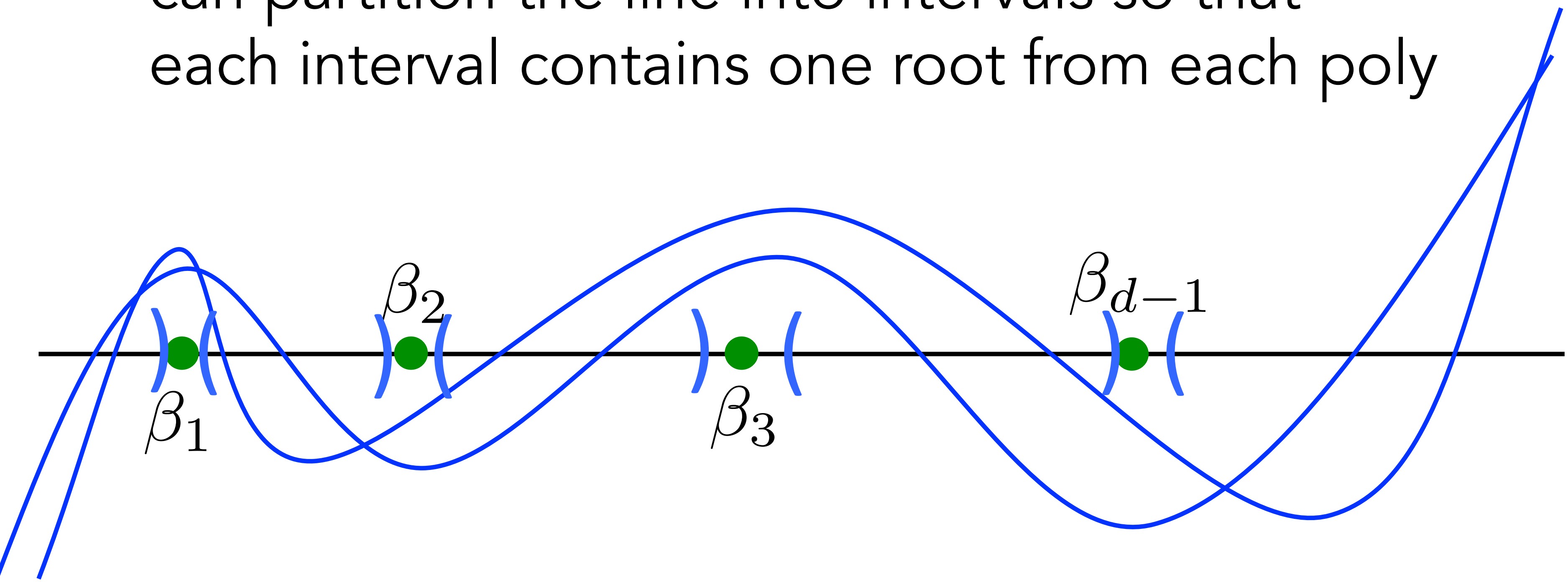
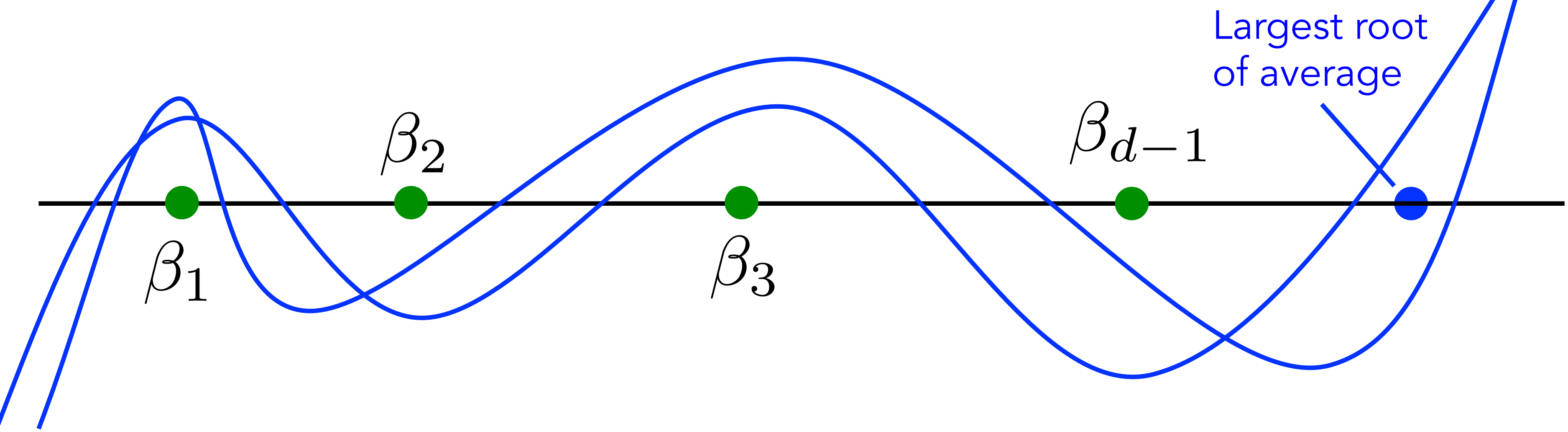


Common Interlacing

If p_1 and p_2 have a common interlacing,

$$\text{max-root}\,(p_i) \leq \text{max-root}\,(\mathbb{E}_i\,[\,p_i\,])$$

for some i.



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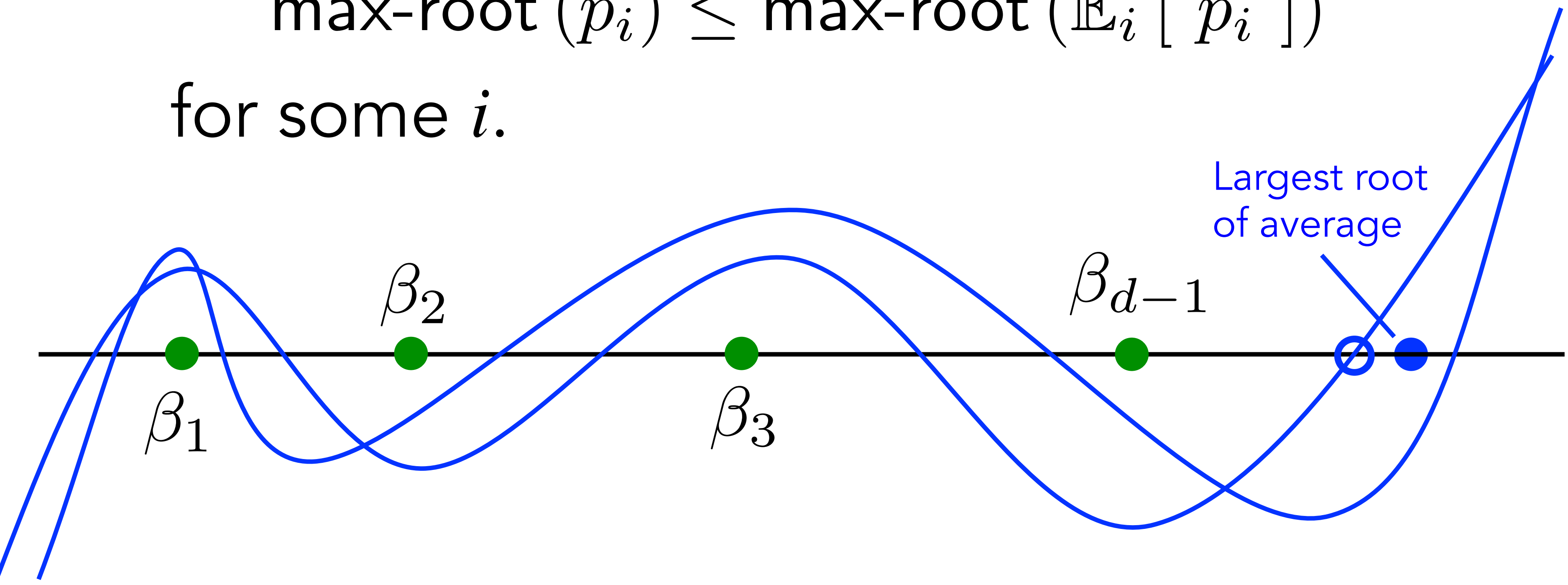
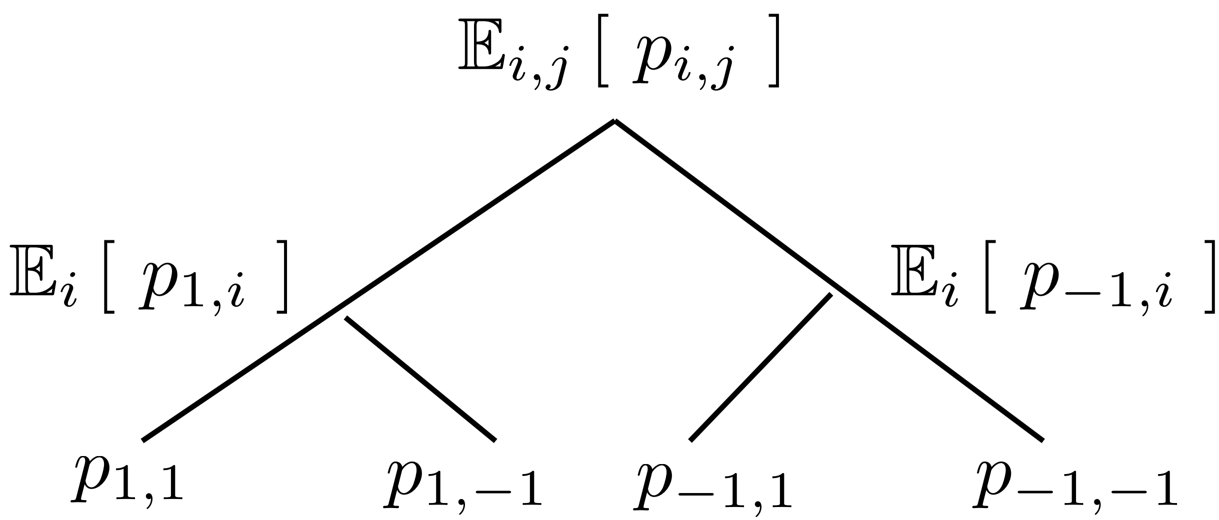


Interlacing Family of Polynomials

$\{p_s\}_{s \in \{\pm 1\}^m}$ is an interlacing family

If the polynomials can be placed on the leaves of a tree so that when put average of descendants at nodes siblings have common interlacings



Interlacing Family of Polynomials

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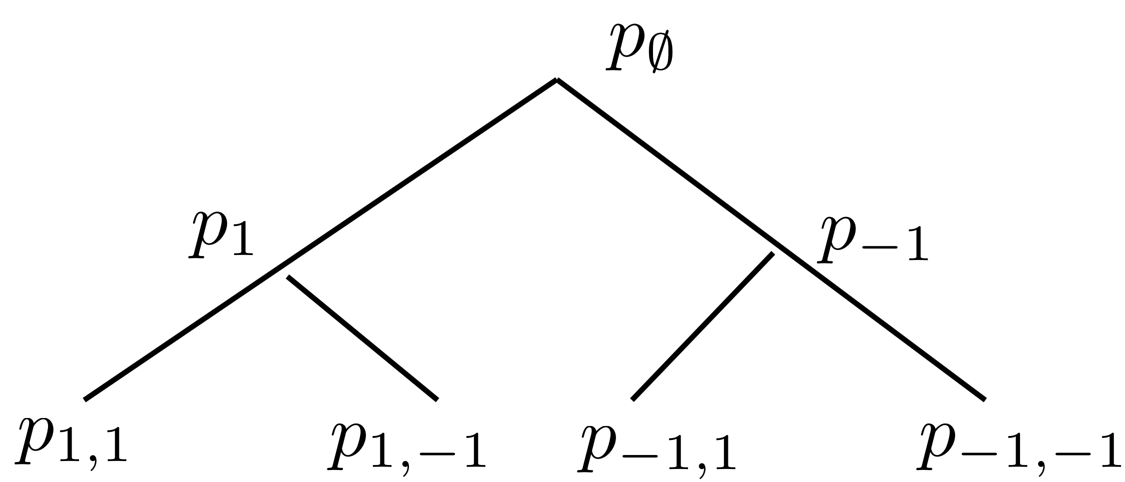
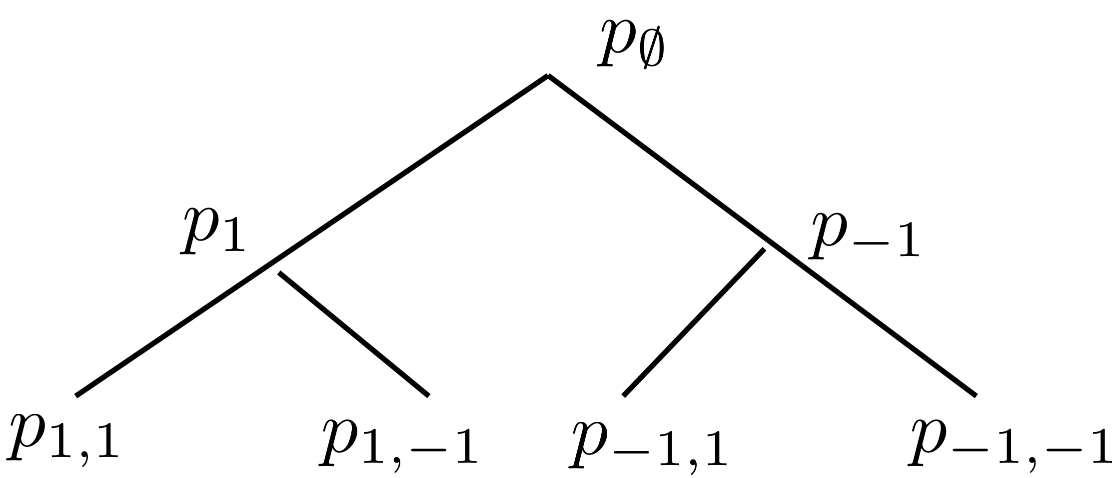


Interlacing Family of Polynomials

Theorem:

There is an s so that

$$\text{max-root}\,(p_s(x)) \leq \text{max-root}\left(\mathop{\mathbb{E}}_{s}[\ p_s(x)\ ]\right)$$



An interlacing family

Theorem:

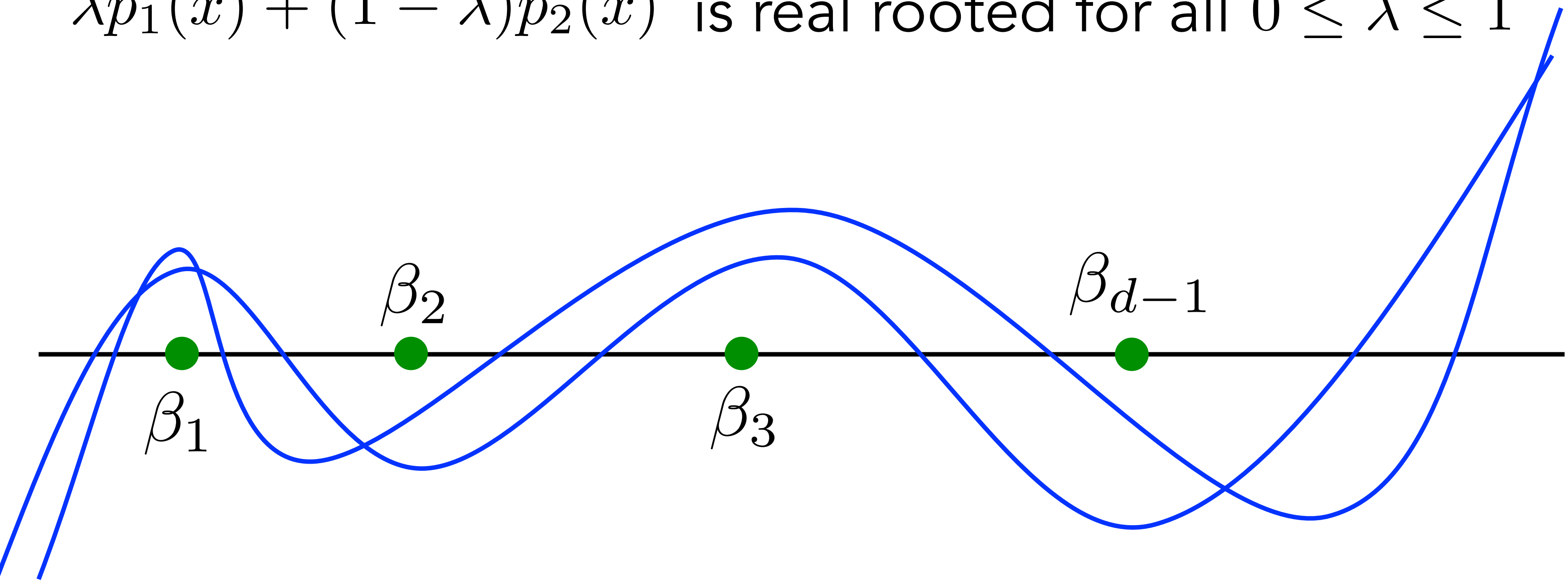
Let $p_s(x) = \chi_{A_s}(x)$

$\{p_s\}_{s \in \{\pm 1\}^m}$ is an interlacing family

Interlacing

$p_1(x)$ and $p_2(x)$ have a common interlacing iff

$\lambda p_1(x) + (1-\lambda)p_2(x)$ is real rooted for all $0 \leq \lambda \leq 1$



To prove interlacing family

Let $p_{s_1,...,s_k}(x) = \mathbb{E}_{s_{k+1},...,s_m} [\ p_{s_1,...,s_m}(x)\ ]$

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$s_1, \ldots, s_k$ *are fixed*

s_{k+1} *is* 1 *with probability* λ, *-*1 *with* $1-\lambda$

$s_{k+2}, \ldots, s_m$ *are uniformly* ± 1

Generalization of Heilmann-Lieb

We prove

$$\mathop{\mathbb{E}}_{s \in \{\pm 1\}^m} [\ p_s(x)\ ] \quad \text{is real rooted}$$

for every independent distribution on the entries of s

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Mixed Characteristic Polynomials

For $a_1, ..., a_n$ independently chosen random vectors

$$\mathbb{E}\left[\ \mathrm{poly}(\textstyle\sum_i a_i a_i^T)\ \right] = \mu(A_1, ..., A_n)$$

is their *mixed characteristic polynomial*.

Theorem: Mixed characteristic polynomials are real rooted.

Proof: Using theory of real stable polynomials.

Mixed Characteristic Polynomials

For $a_1, \ldots, a_n$ independently chosen random vectors

$$\mathbb{E}\left[\ \mathrm{poly}(\textstyle\sum_i a_i a_i^T)\ \right] = \mu(A_1, \ldots, A_n)$$

is their *mixed characteristic polynomial.*

Obstacle: our matrix is a sum of random rank-2 matrices

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \quad \text{or} \quad \begin{matrix} 0 & -1 \\ -1 & 0 \end{matrix}$$

Mixed Characteristic Polynomials

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is their *mixed characteristic polynomial.*

Solution: add to the diagonal

1	1	or	1	-1
1	1		-1	1

Generalization of Heilmann-Lieb

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Implies $\chi_{A_s}(x)$ is an interlacing family

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Implies $\chi_{A_s}(x)$ is an interlacing family

Conclude there is an s so that

$$\text{max-root}\left(\chi_{A_s}(x)\right) \leq 2\sqrt{d-1}$$

Universal Covers

The universal cover of a graph G
is a tree T of which G is a quotient.

vertices map to vertices
edges map to edges
homomorphism on neighborhoods

Is the tree of non-backtracking walks in G.

The universal cover of a d-regular graph
is the infinite d-regular tree.

Quotients of Trees

d-regular Ramanujan as
quotient of infinite d-ary tree

Spectral radius and norm of inf d-ary tree are

$$2\sqrt{d-1}$$

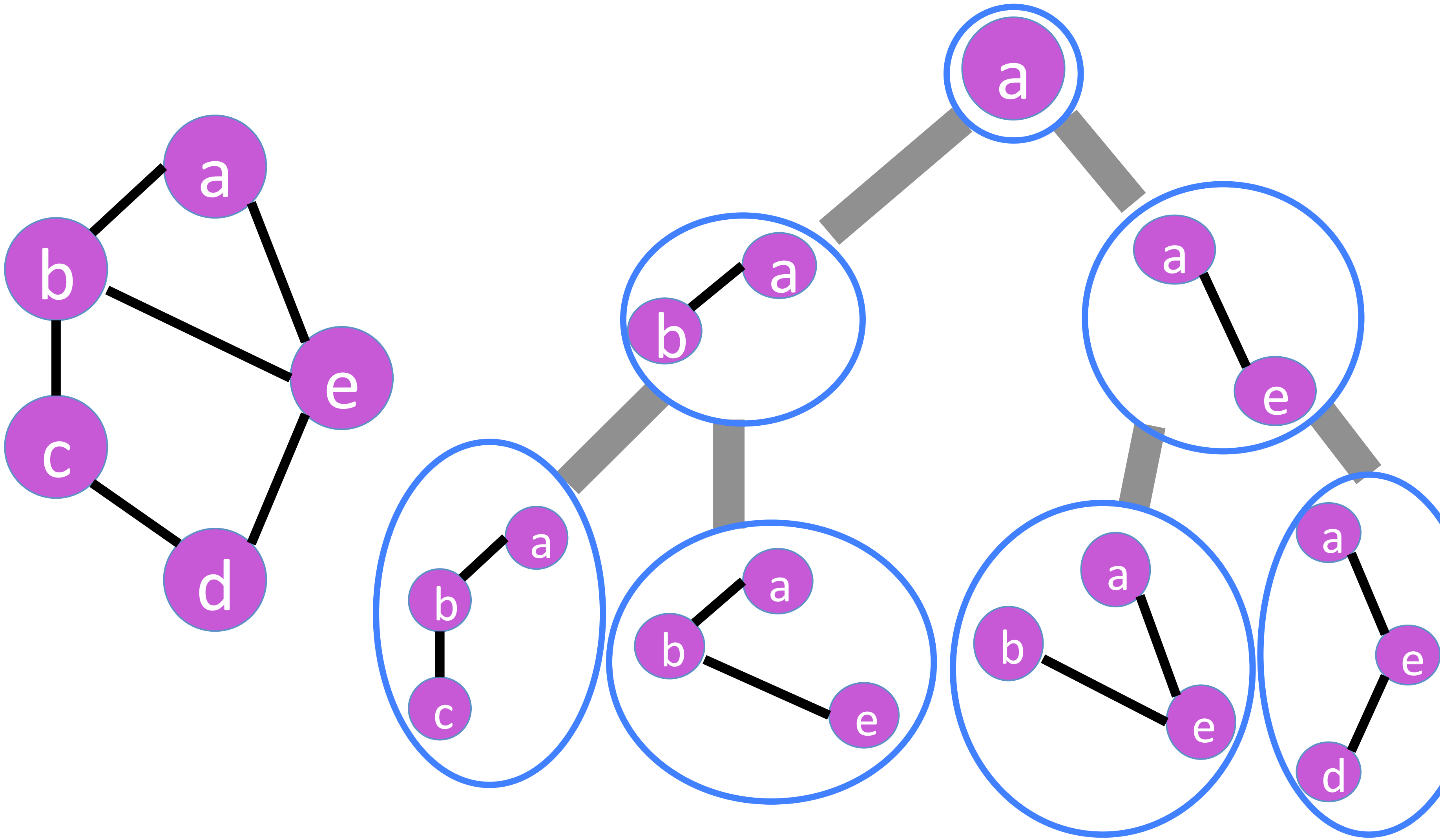
Godsil's Proof of Heilmann-Lieb

$T(G,v)$: the path tree of G at v

vertices are paths in G starting at v

edges to paths differing in one step

Godsil's Proof of Heilmann-Lieb



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Theorem:

The matching polynomial divides the characteristic polynomial of $T(G,v)$

Godsil's Proof of Heilmann-Lieb

$T(G,v)$: the path tree of G at v
vertices are paths in G starting at v
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Theorem:
The matching polynomial divides
the characteristic polynomial of $T(G,v)$

Is a subgraph of infinite tree,
so has smaller spectral radius

Quotients of Trees

(c,d)-biregular bipartite Ramanujan as quotient of infinite *(c,d)*-ary tree

Spectral radius $\sqrt{d-1}+\sqrt{c-1}$

For *(c,d)*-regular bipartite Ramanujan graphs

$$\sqrt{d-1}+\sqrt{c-1}$$

Questions

Non-bipartite Ramanujan Graphs of every degree?

Efficient constructions?

Explicit constructions?