Spectral Sparsification of Graphs and Approximations of Matrices

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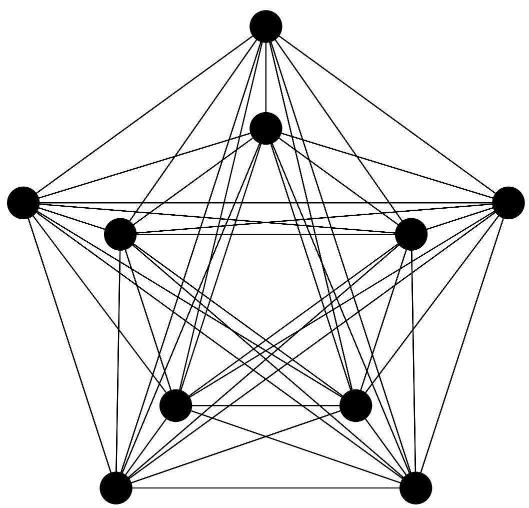

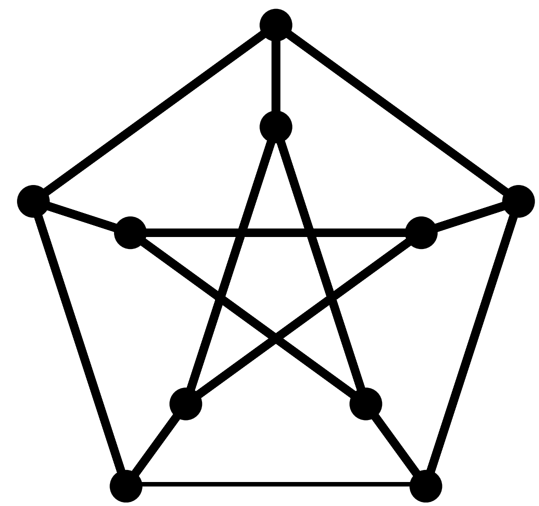

joint work with
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Nikhil Srivastava (IAS/Princeton)
Shang-Hua Teng (USC)

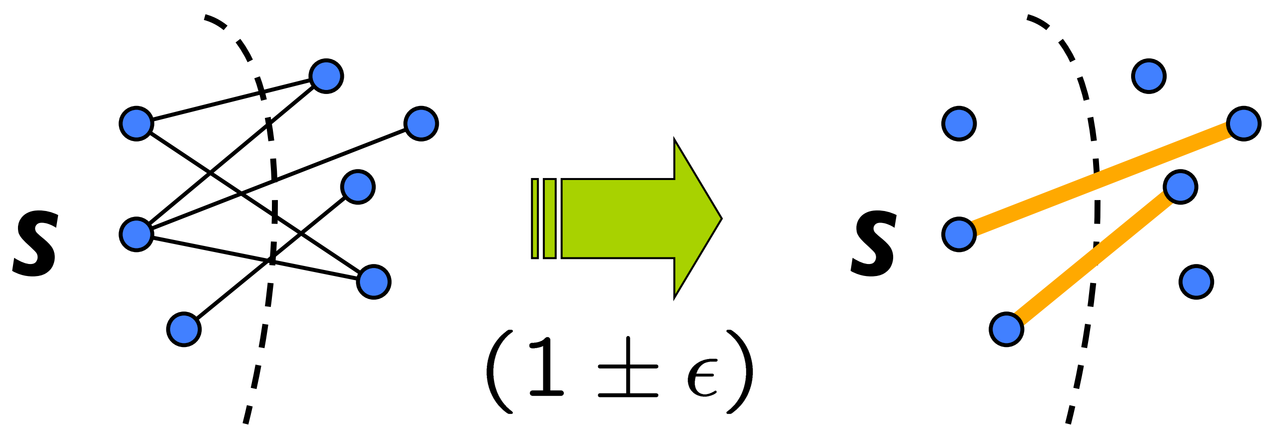
Toronto, Sep 30, 2011

Objective of Sparsification:

Approximate any (weighted) graph by a sparse weighted graph.

Spanners - Preserve Distances [Chew '86]

Cut-Sparsifiers – preserve wt of edges leaving every set $S \subseteq V$ [Benczur-Karger '96]



Spectral Sparsification [S-Teng]

Approximate any (weighted) graph by a sparse weighted graph.

Graph ⟷ Quadratic Form

$$G = (V, E, w)$$

$$\sum_{(u,v)\in E} w_{u,v}(x(u) - x(v))^2$$

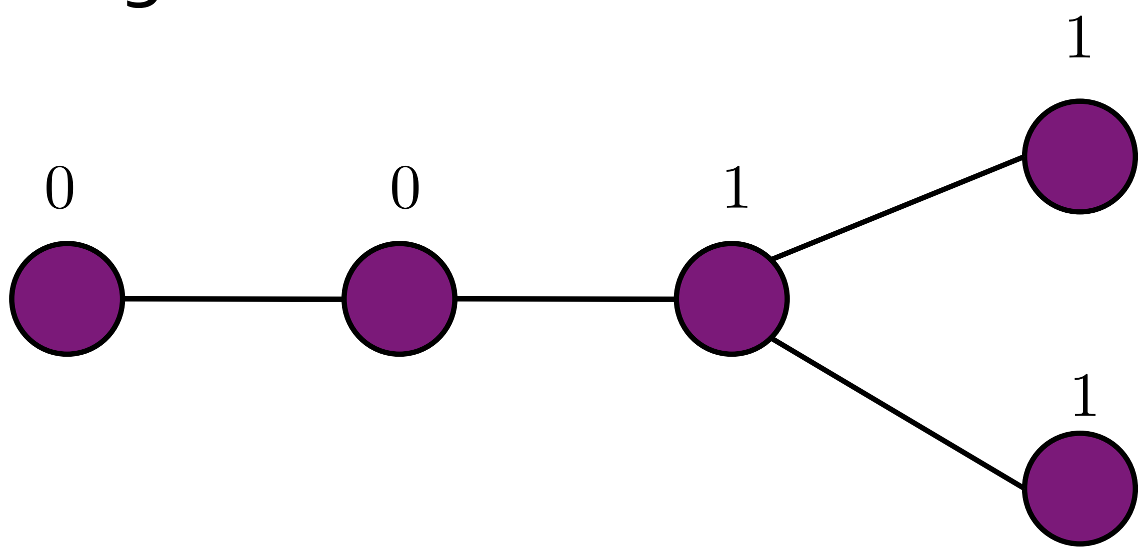
$$= x^T L_G x$$

L_G is Laplacian of G

Laplacian Quadratic Form, examples

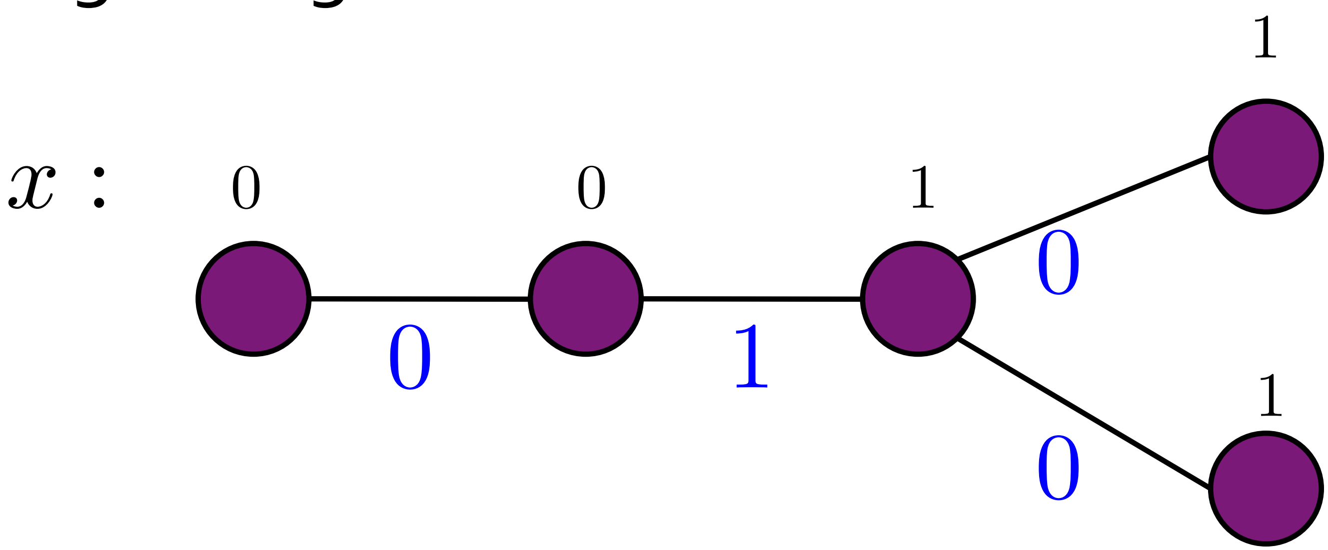
All edge-weights are 1

$x:$



Laplacian Quadratic Form, examples

All edge-weights are 1



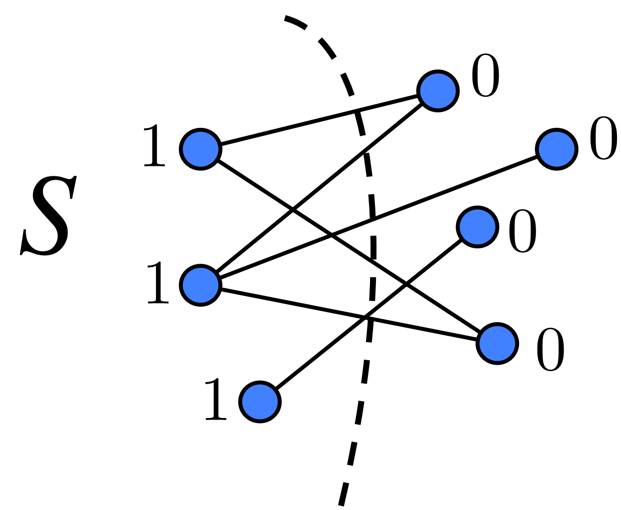
$$x^T L_G x =$$ Sum of squares of differences across edges

$$= 1$$

Laplacian Quadratic Form, examples

When x is the characteristic vector of a set S,
sum the weights of edges on the boundary of S

$$x^T L_G x = \sum_{\substack{(u,v) \in E \\ u \in S \\ v \notin S}} w_{u,v}$$



Laplacian Matrices (quick review)

$$x^T L_G x = \sum_{(u,v) \in E} w_{u,v} (x(u) - x(v))^2$$

$$L_G = D_G - A_G$$

Positive semi-definite

If connected, nullspace = Span($\mathbf{1}$)

Laplacian Matrices

$$x^T L_G x = \sum_{(u,v)\in E} w_{u,v}(x(u) - x(v))^2$$

$$L_G = \sum_{(u,v)\in E} w_{u,v} L_{u,v}$$

E.g. $$L_{1,2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}$$

Laplacian Matrices

$$x^T L_G x = \sum_{(u,v)\in E} w_{u,v}(x(u) - x(v))^2$$

$$L_G = \sum_{(u,v)\in E} w_{u,v} L_{u,v}$$

$$= \sum_{(u,v)\in E} w_{u,v}(b_{u,v} b_{u,v}^T)$$

Sum of outer products

where $\quad b_{u,v} = \delta_u - \delta_v$

Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$G \preccurlyeq H$$

Iff, for all $x \in R^V$

$$x^T L_G x \leq x^T L_H x$$

Inequalities on Graphs

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

$$G \preccurlyeq k \cdot H \qquad \textit{(multiply edge weights by k)}$$

Iff, for all $x \in R^V$

$$x^T L_G x \leq k \cdot x^T L_H x$$

Approximation

For graphs $G = (V, E, w)$ and $H = (V, F, z)$

H is an ϵ-approximation of G if

$$(1+\epsilon)^{-1} G \preccurlyeq H \preccurlyeq (1+\epsilon) G$$

That is, for all $x \in R^V$

$$\frac{1}{1+\epsilon} \leq \frac{\sum_{(u,v)\in F} z_{u,v} (x(u) - x(v))^2}{\sum_{(u,v)\in E} w_{u,v} (x(u) - x(v))^2} \leq 1+\epsilon$$

Implications of Approximation

$$(1+\epsilon)^{-1}G \preccurlyeq H \preccurlyeq (1+\epsilon)G$$

L_H and L_G have similar eigenvalues

$$(1+\epsilon)^{-1}x^T L_H^+ x \leq x^T L_G^+ x \leq (1+\epsilon)x^T L_H^+ x$$

Effective resistances between vertices similar.

L_H is a good preconditioner for L_G

Spectral Sparsification [S-Teng]

For an input graph G with n vertices,

find a sparse graph H having $\tilde{O}(n)$ edges

so that H is an ϵ-approximation of G

Expanders Approximate Complete Graphs

Strong expanders:

d-regular graphs on n vertices

for $x \perp \mathbf{1}$, $x^T L_H x \sim d x^T x$

Expanders Approximate Complete Graphs

For H a d-regular strong expander,

For $x \perp \mathbf{1}$, $\|x\| = 1$ $\quad x^T L_H x \sim d$

For G the complete graph on n verts.

all non-zero eigs of L_G are n.

For $x \perp \mathbf{1}$, $\|x\| = 1$ $\quad x^T L_G x = n$

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For $x \perp \mathbf{1}$, $\|x\| = 1$ $\qquad x^T L_G x = n$

$\frac{n}{d} H$ is a good approximation of G

Best Approximations of Complete Graphs

Ramanujan Expanders
[Margulis, Lubotzky-Phillips-Sarnak]

$$d - 2\sqrt{d-1} \leq \lambda(L_H) \leq d + 2\sqrt{d-1}$$

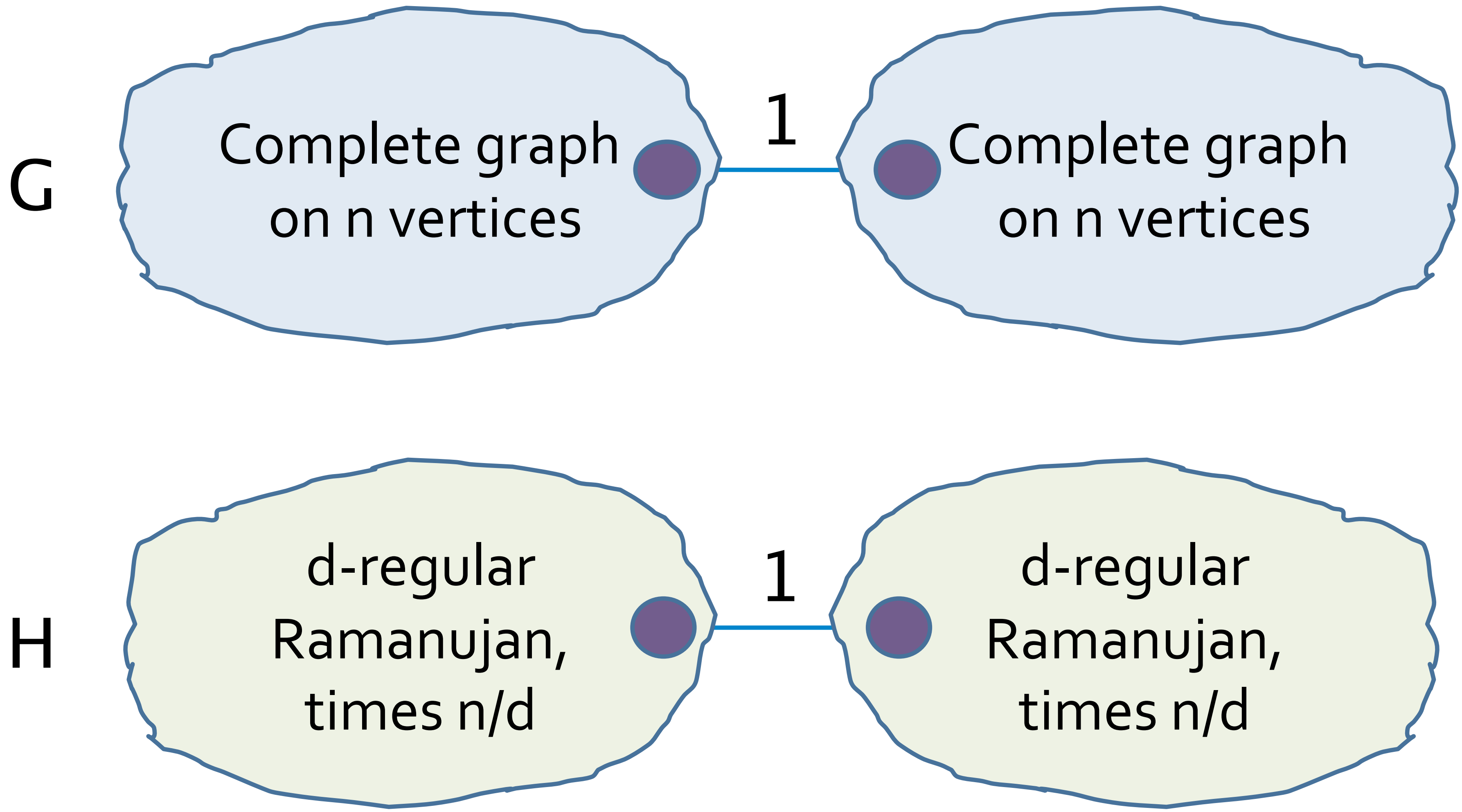
Best Approximations of Complete Graphs

Ramanujan Expanders
[Margulis, Lubotzky-Phillips-Sarnak]

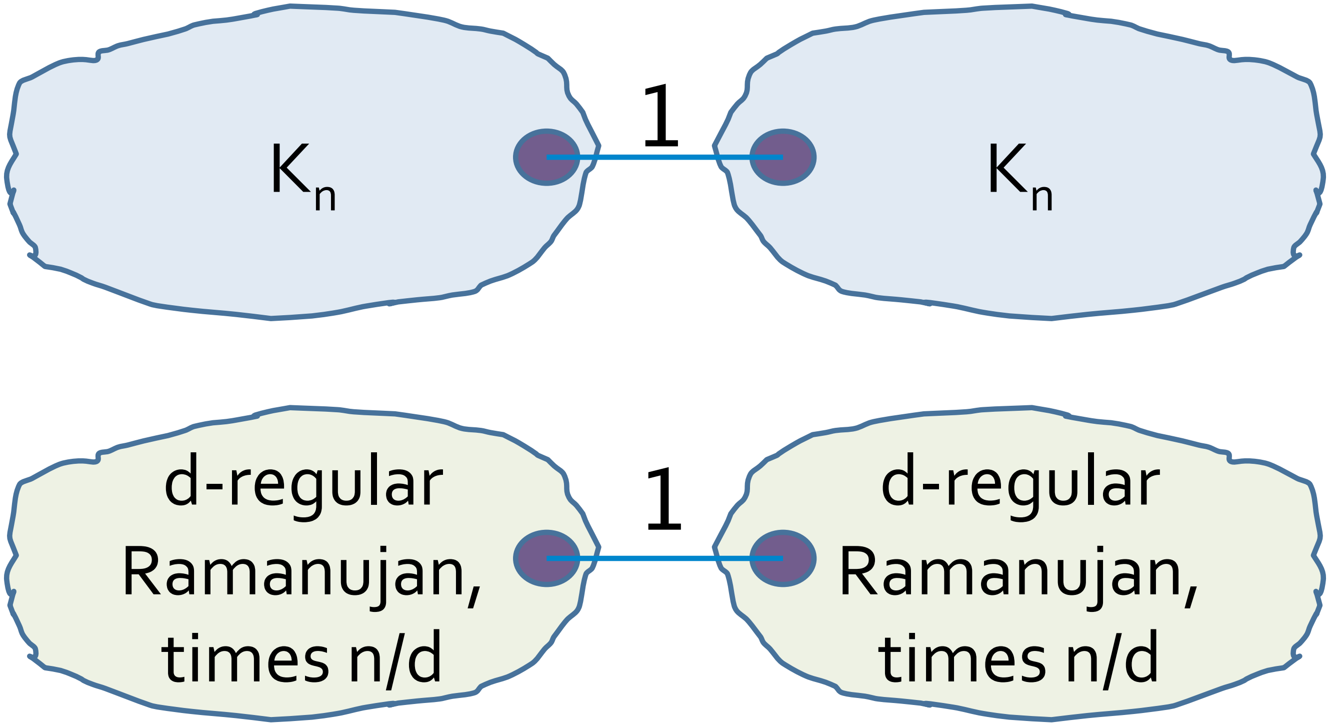
$$d - 2\sqrt{d-1} \leq \lambda(L_H) \leq d + 2\sqrt{d-1}$$

Can we approximate every graph this well?

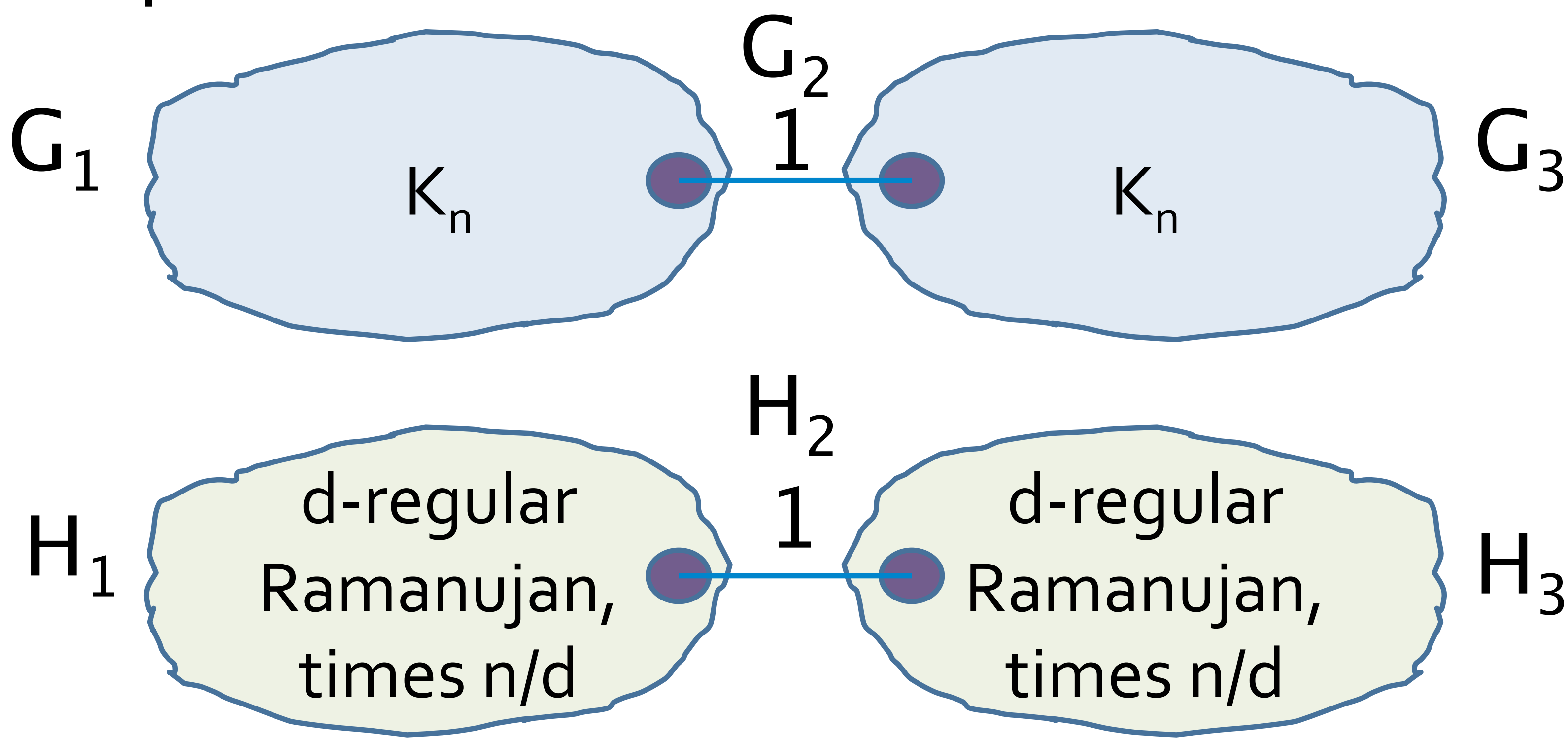
Example: Dumbbell



Example: Dumbbell



Example: Dumbbell



$$G = G_1 + G_2 + G_3$$
$$H = H_1 + H_2 + H_3$$

$$G_1 \preccurlyeq (1+\epsilon)H_1$$
$$G_2 = H_2$$
$$G_3 \preccurlyeq (1+\epsilon)H_3$$

$$G \preccurlyeq (1+\epsilon)H$$

Main Theorem

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

1. H is an ϵ -approximation of G

2. $|F| \leq |V| \, (2+\epsilon)^2/\epsilon^2$ (Batson-S-Srivastava 09)

3. F ⊆ E

Main Theorem (today)

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

1. H is a 2.6-approximation of G
2. $|F| \leq 6\,|V|$ (Batson-S-Srivastava 09)
3. F ⊆ E

Reduction to Matrix Approximation

$$1 \leq \frac{x^T L_H x}{x^T L_G x} \leq 13 \qquad \forall x \perp \mathbf{1}$$

Setting $x = L_G^{-1/2} z$

Becomes $$1 \leq \frac{z^T L_G^{-1/2} L_H L_G^{-1/2} z}{z^T z} \leq 13$$

$$\forall z \perp \mathbf{1}$$

Reduction to Matrix Approximation

Suffices to show

$$1 \leq \frac{z^T L_G^{-1/2} L_H L_G^{-1/2} z}{z^T z} \leq 13 \qquad \forall z \perp \mathbf{1}$$

$$1 \leq \lambda \left(L_G^{-1/2} L_H L_G^{-1/2} \right) \leq 13$$

Reduction to Matrix Approximation

$$1 \leq \lambda\left(L_G^{-1/2} L_H L_G^{-1/2}\right) \leq 13$$

Write $L_H = \sum s_e (b_e b_e^T)$

weight of edge e in graph H

Laplacian of edge e, as outer-product of vectors

Need $1 \leq \lambda(\sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2}) \leq 13$

At most $6n$ of the s_e non-zero

Reduction to Matrix Approximation

Need $$1 \leq \lambda\Big(\sum_{e} s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2}\Big) \leq 13$$

At most $6n$ of the s_e non-zero

Reduction to Matrix Approximation

Need $$1 \leq \lambda\Big(\sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2}\Big) \leq 13$$

At most $6n$ of the s_e non-zero

Recall $$L_G = \sum_e b_e b_e^T$$

So $$\sum_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} = I$$

Reduction to Matrix Approximation

Need $$1 \leq \lambda\Big(\sum_e s_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} \Big) \leq 13$$

At most $6n$ of the s_e non-zero

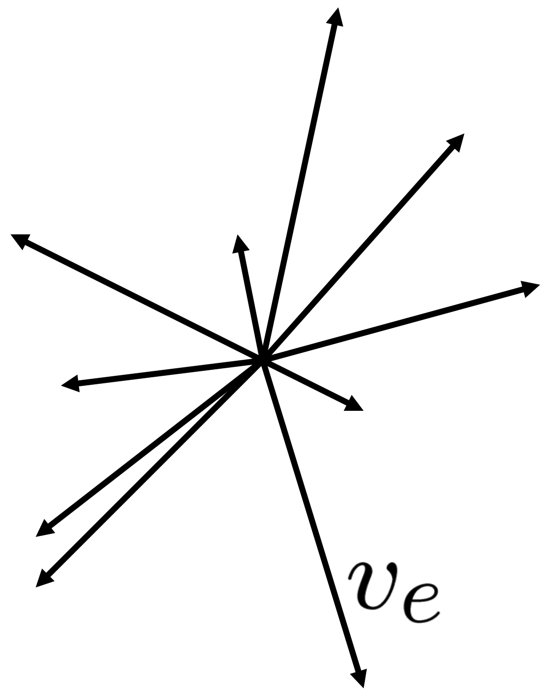
Set $v_e = L_G^{-1/2} b_e$

$$1 \leq \lambda\Big(\sum_e s_e v_e v_e^T \Big) \leq 13$$

$$\sum_e v_e v_e^T = \sum_e L_G^{-1/2} b_e b_e^T L_G^{-1/2} = L_G^{-1/2} L_G L_G^{-1/2} = id$$

A closer look at v_e

$$v_e = L_G^{-1/2} b_e.$$



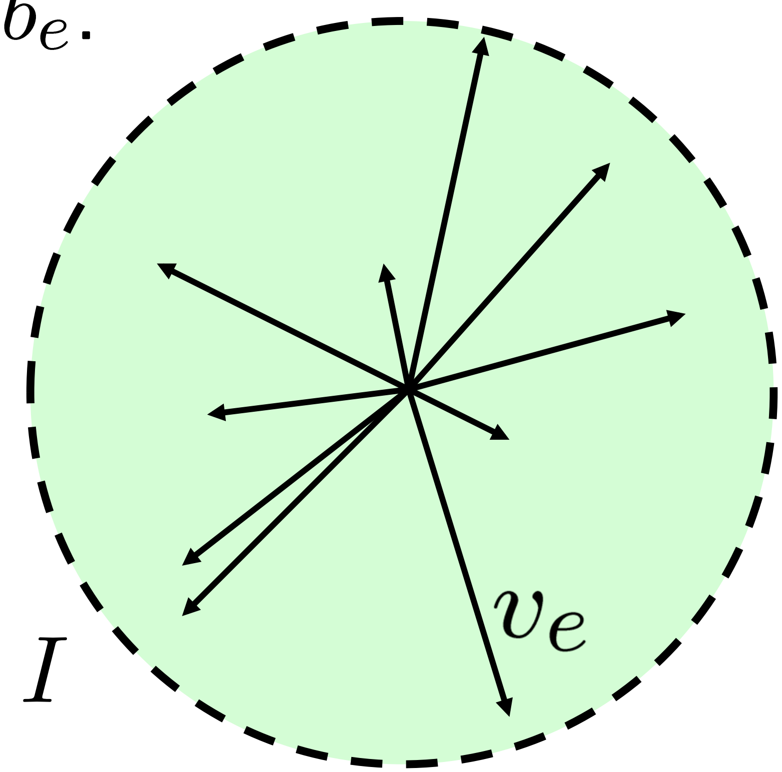
m vectors in ***R***$^{n-1}$

$$\sum_e v_e v_e^T = I$$

"decomposition of identity"

A closer look at v_e

$$v_e = L_G^{-1/2} b_e.$$



m vectors in $\boldsymbol{R}^{n-1}$

$$\sum_e v_e v_e^T = I$$

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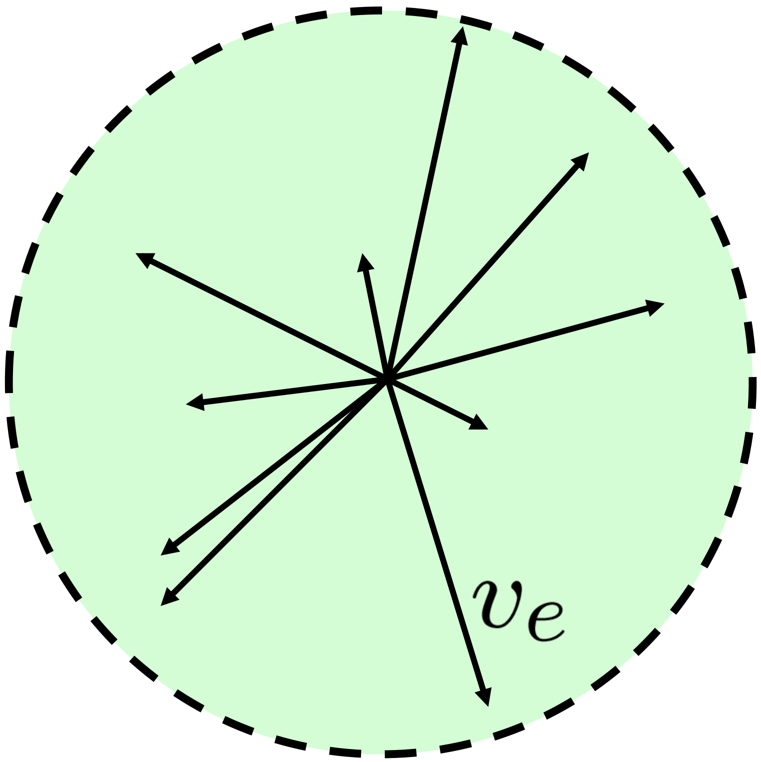
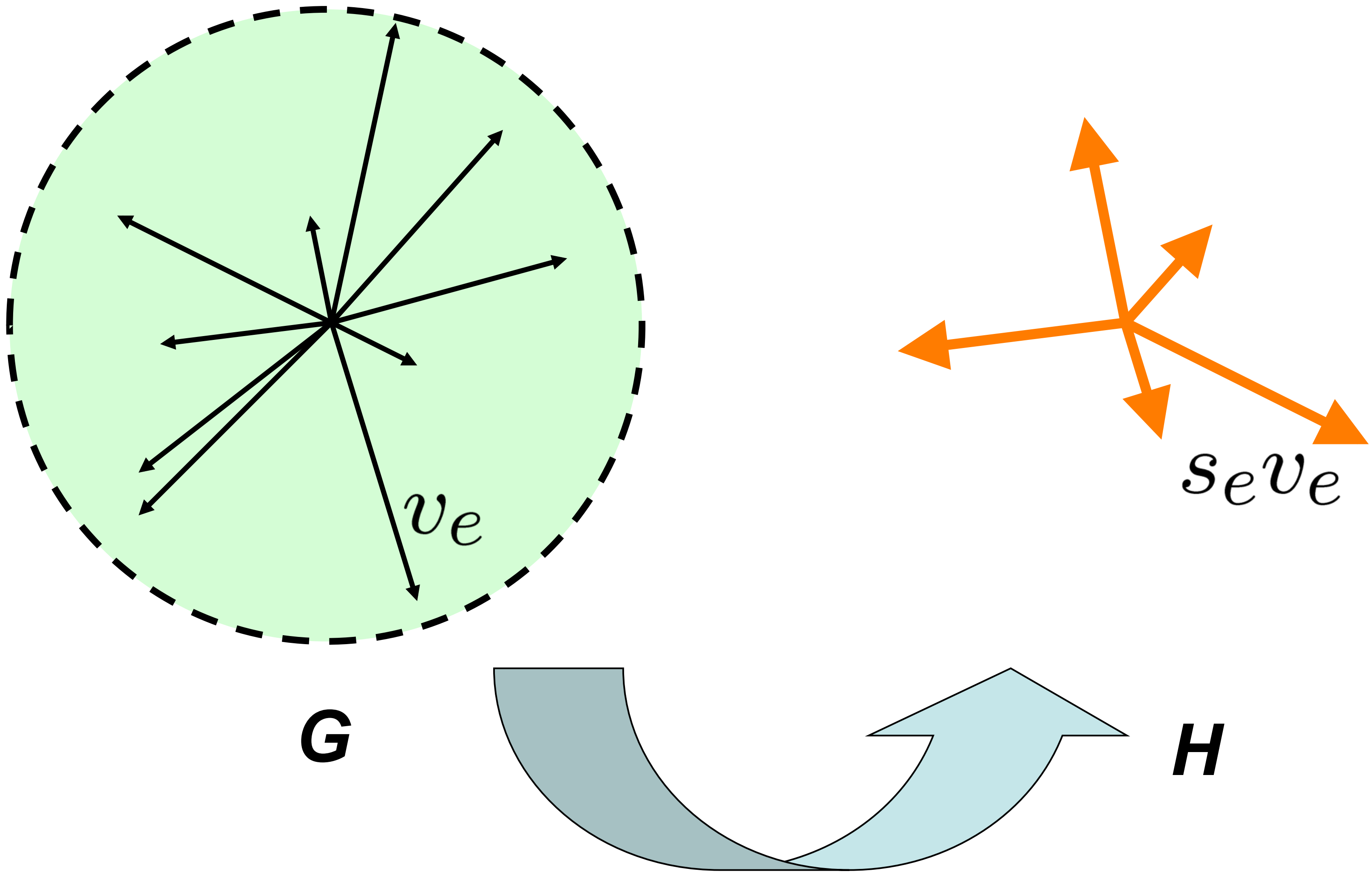
A closer look at v_e



m vectors in $\boldsymbol{R}^{n-1}$

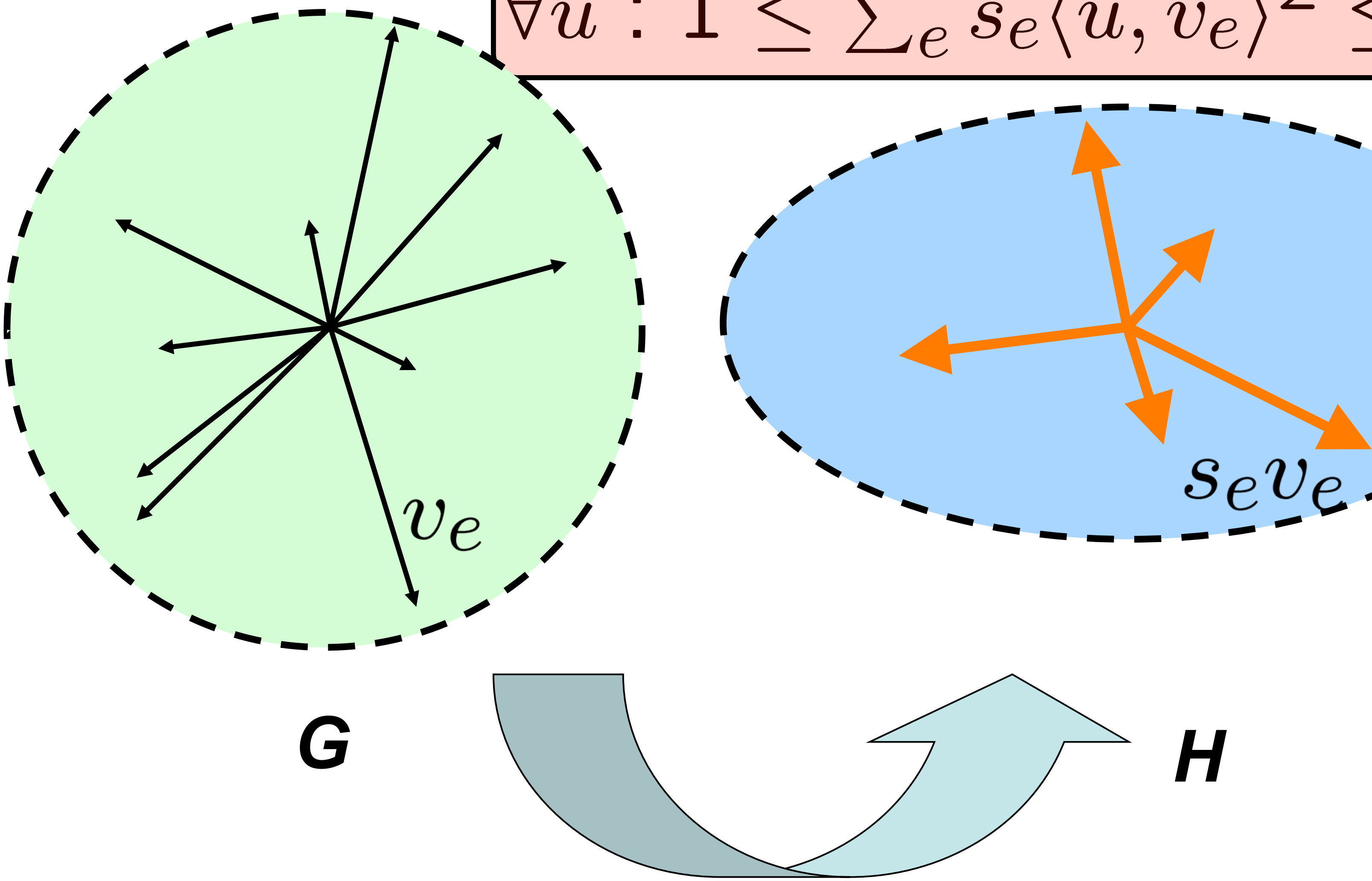
$$\forall u \quad \sum_e \langle u, v_e \rangle^2 = 1$$

Choosing a Subgraph



New Goal

$$\forall u : 1 \leq \sum_e s_e \langle u, v_e \rangle^2 \leq 13$$



Existence theorem

If

$$\sum_e v_e v_e^T = I_n$$

then there are scalars $s_e \geq 0$ with

$$1 \leq \lambda(\sum_e s_e v_e v_e^T) \leq 13$$

and $\left|\{s_e \neq 0\}\right| \leq 6n$.

Existence theorem

If

$$\sum_e v_e v_e^T = I_n$$

then there are scalars $s_e \geq 0$ with

$$1 \leq \lambda\left(\sum_e s_e v_e v_e^T\right) \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$

and $|\{s_e \neq 0\}| \leq dn$

Existence theorem

If

$$\sum_e v_e v_e^T = I_n$$

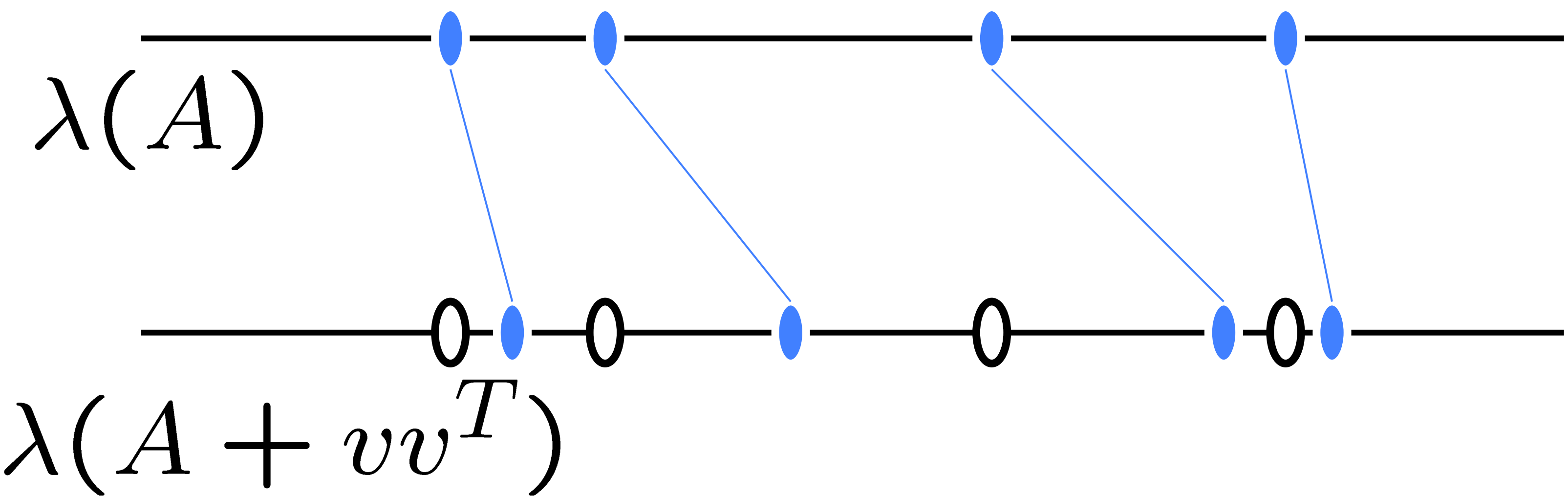
then there are scalars $s_e \geq 0$ with

$$1 \leq \lambda(\sum_e s_e v_e v_e^T) \leq 13$$

and $|\{s_e \neq 0\}| \leq 6n$.

What happens when we add a vector?

Interlacing



More precisely

Characteristic Polynomial:

$$p_A(x) = \det(xI - A)$$

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Rank-one update:

$$p_{A+vv^T} = \left(1 + \sum_i \frac{<v, u_i>^2}{\lambda_i - x}\right) p_A$$

Where $Au_i = \lambda_i u_i$

More precisely

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Rank-one update:

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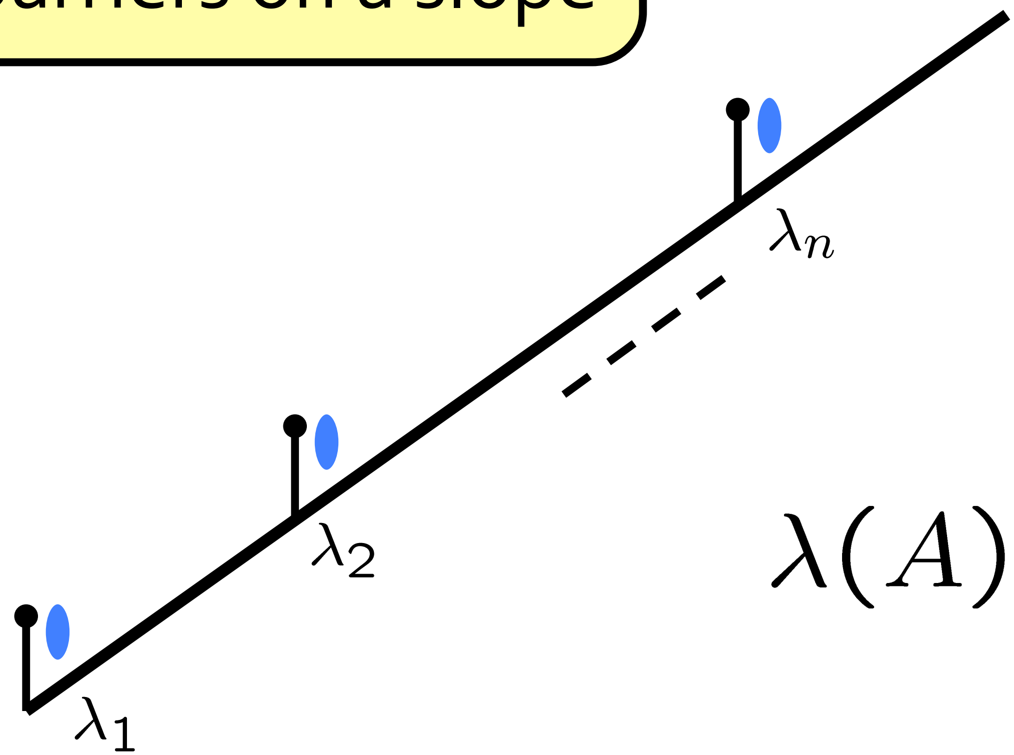
$\lambda(A + vv^T)$ are zeros of ⮥

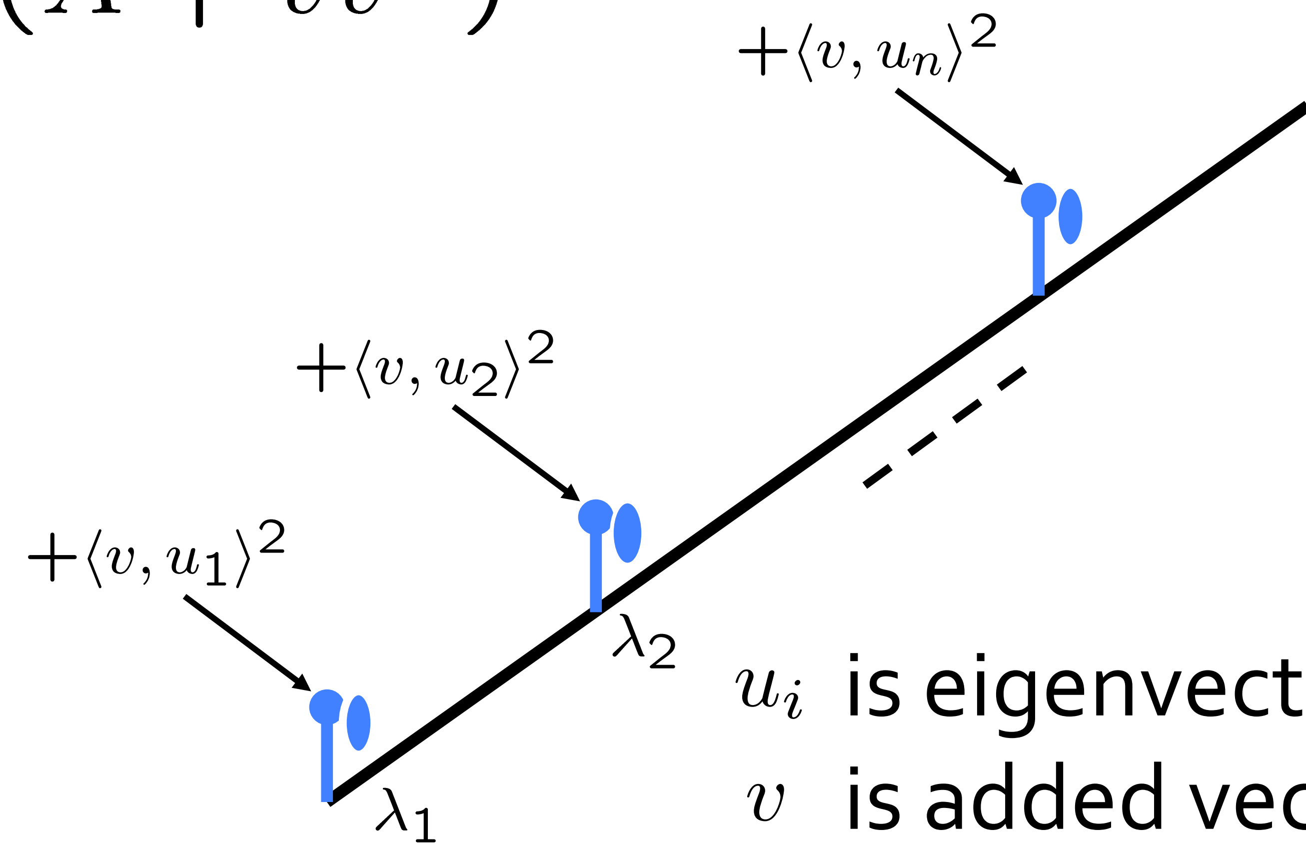
Physical model of interlacing

λ_i = positive unit charges resting at barriers on a slope



Physical model of interlacing

$$\lambda(A + vv^T)$$



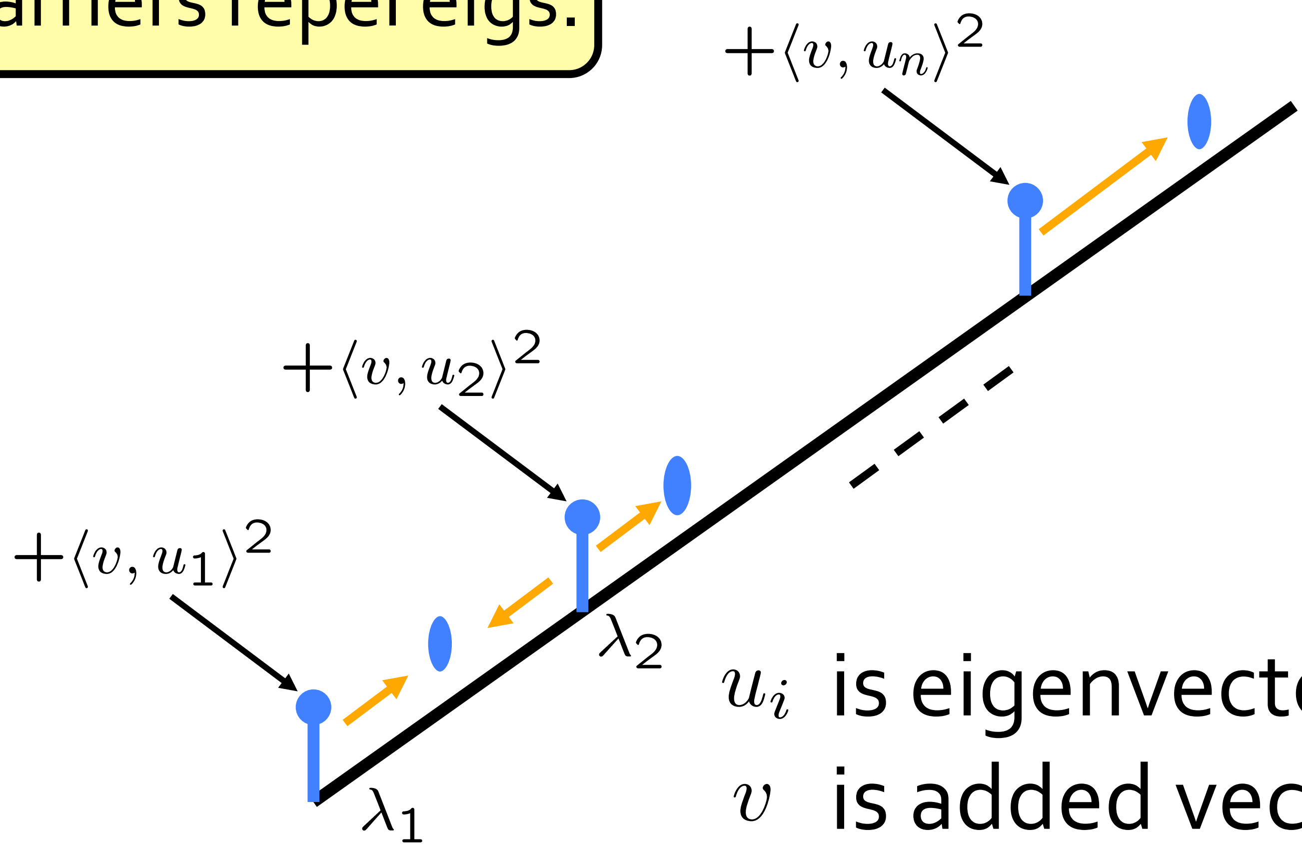
u_i is eigenvector

v is added vector

$\langle v, u_i \rangle^2$ charge on barrier

Physical model of interlacing

Barriers repel eigs.



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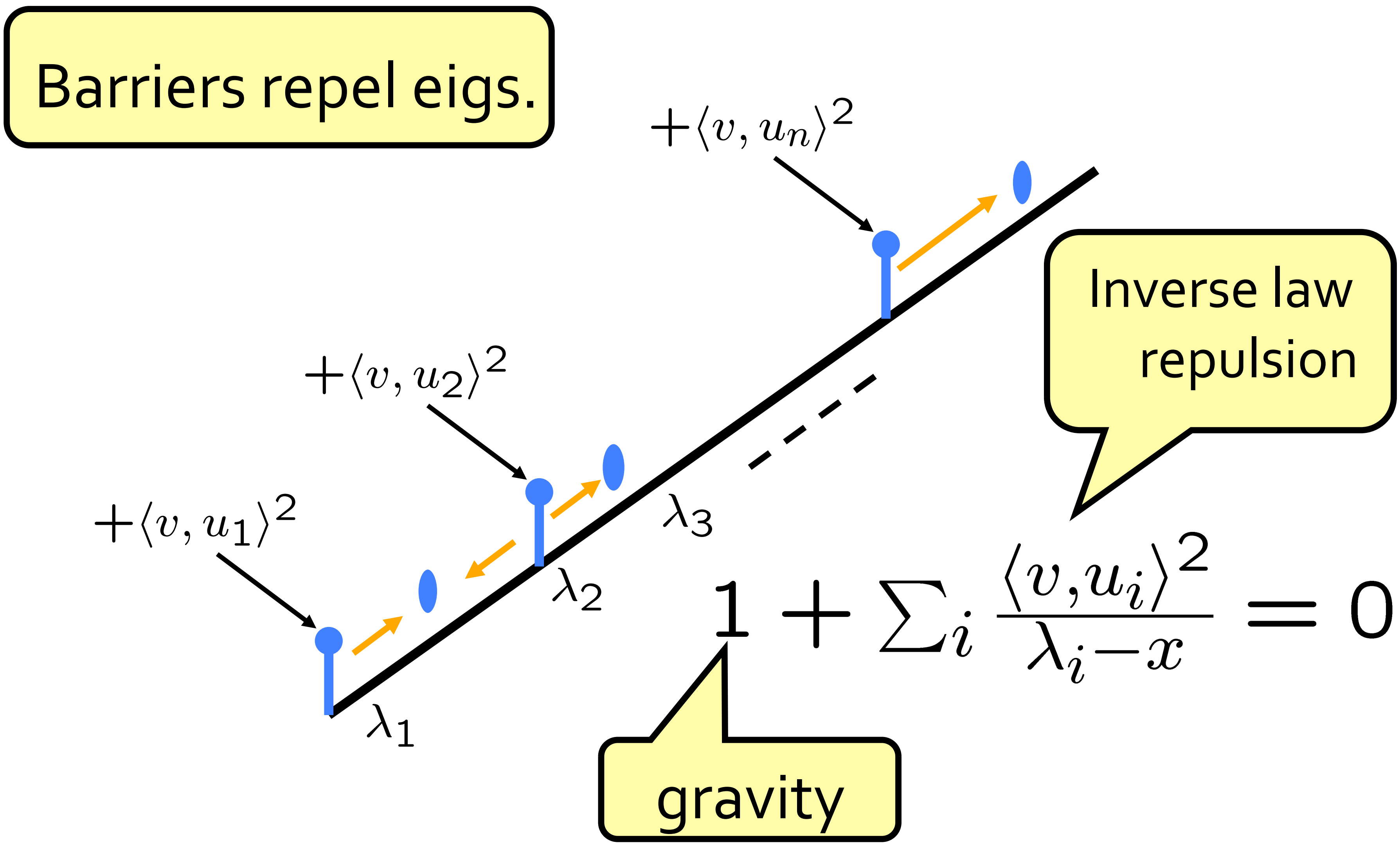
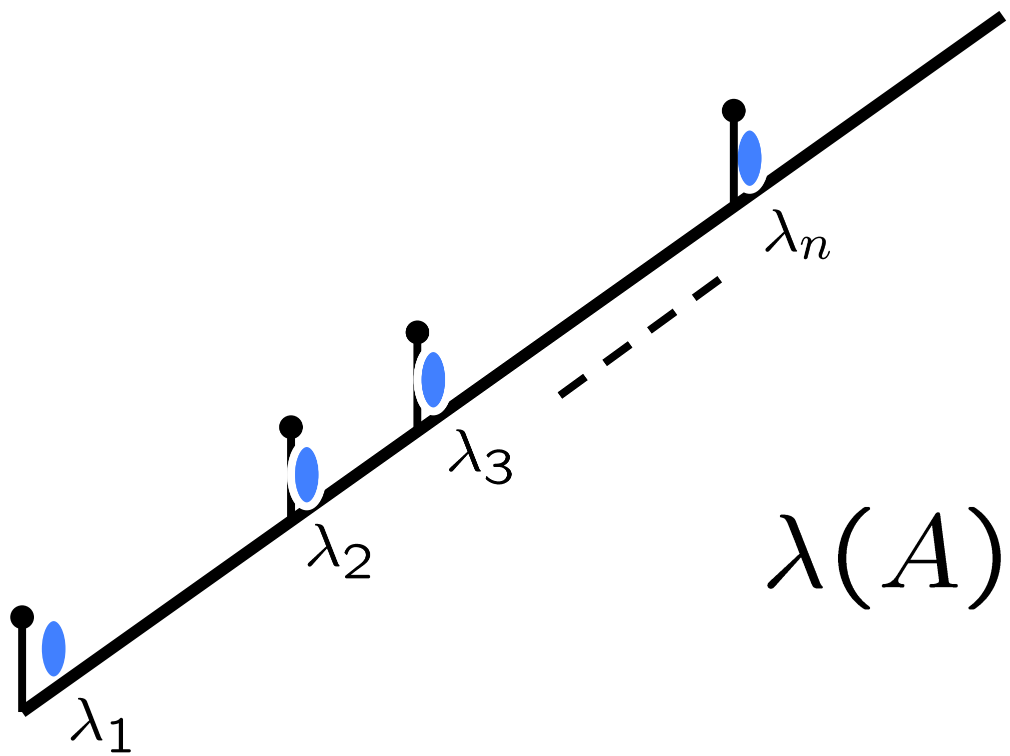
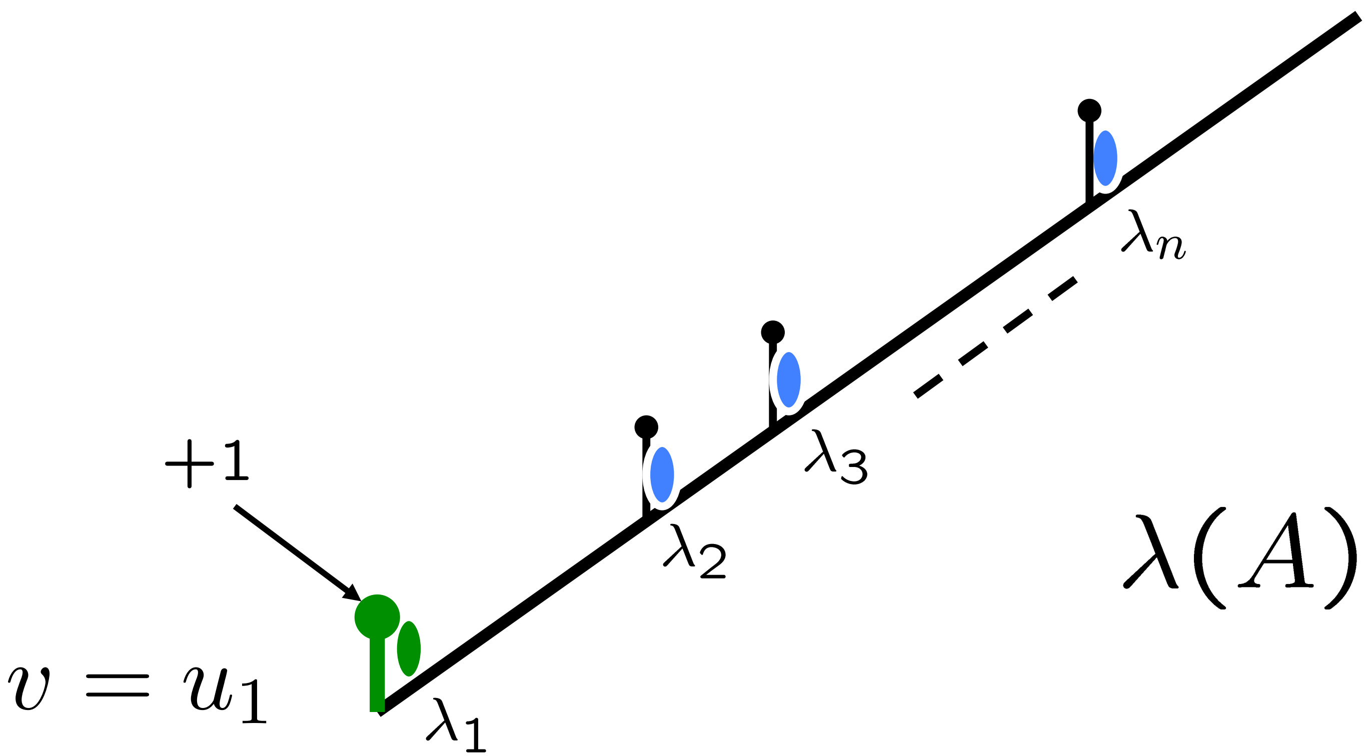


Examples



Ex1: All weight on u_1



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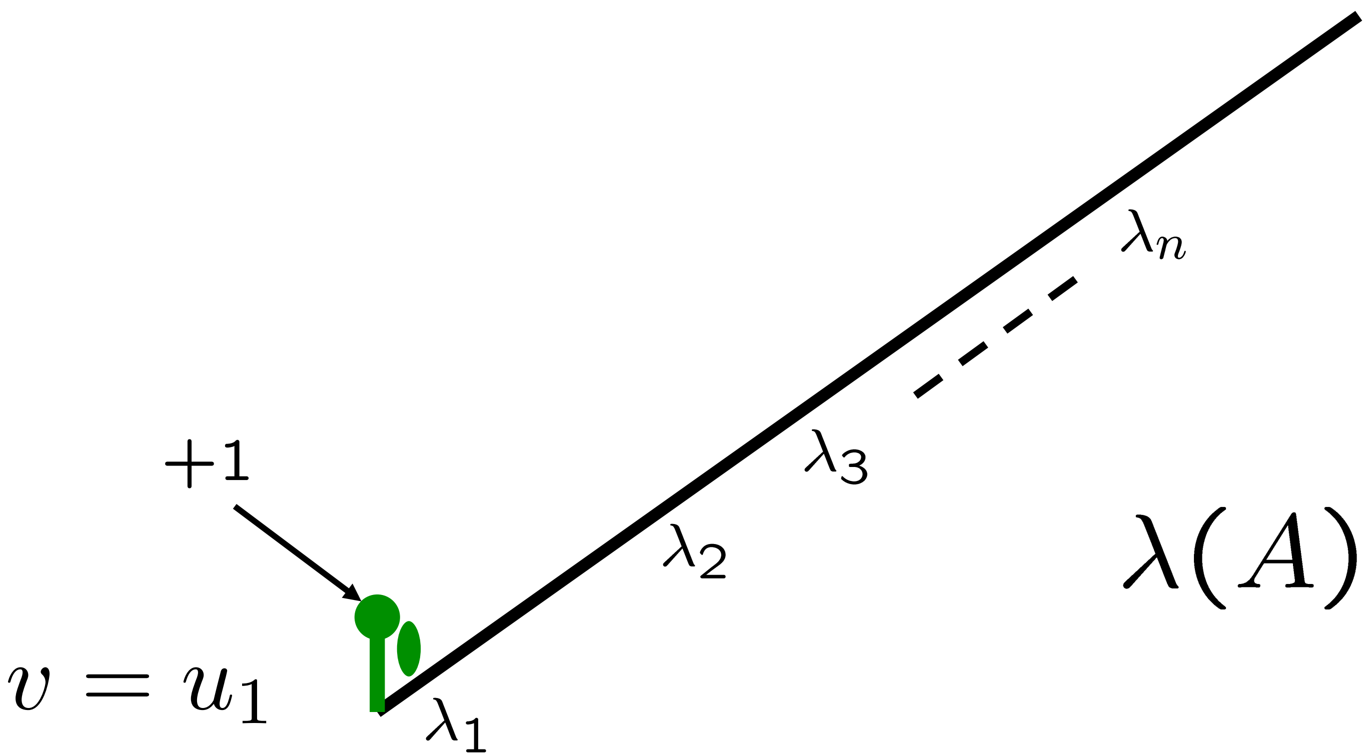

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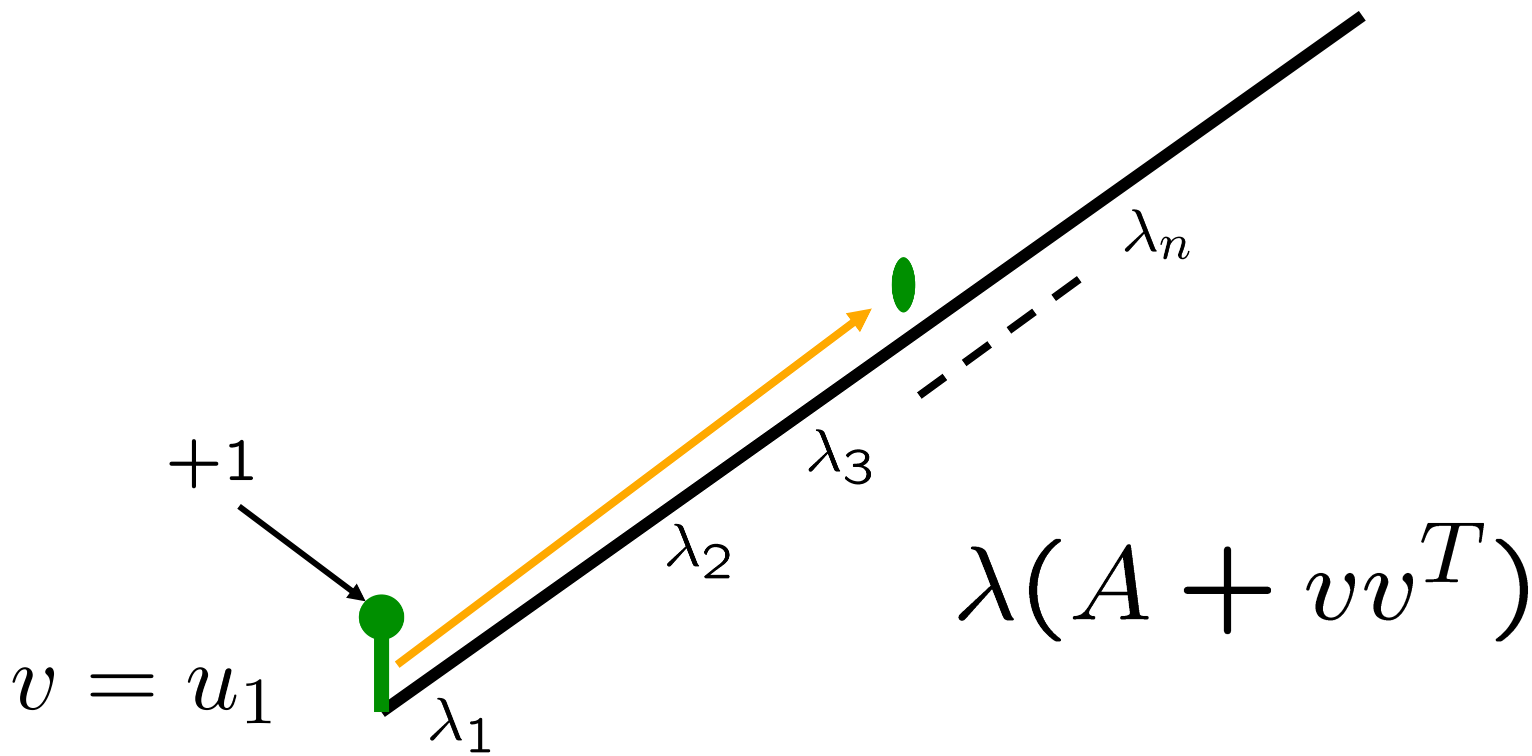

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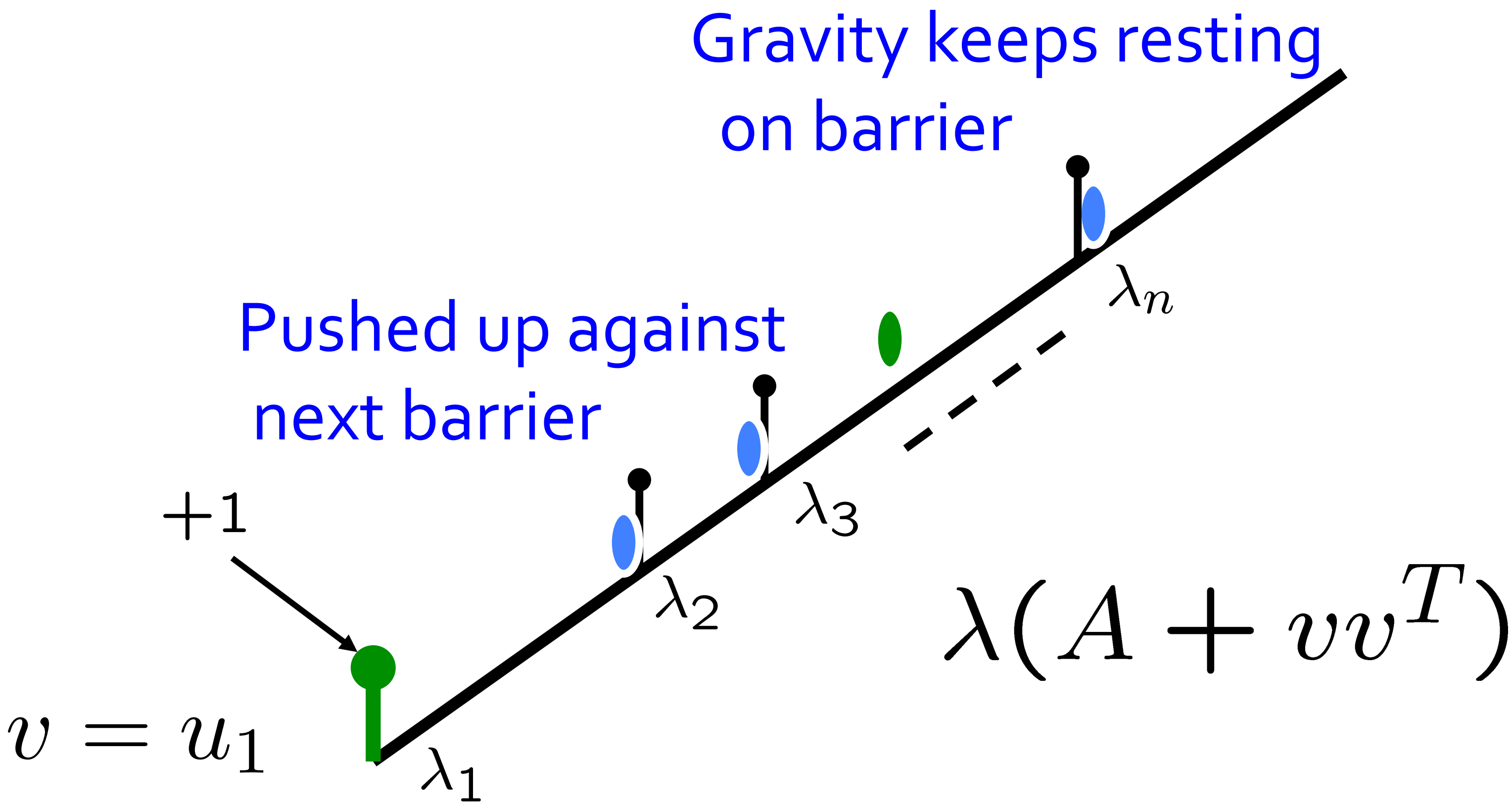
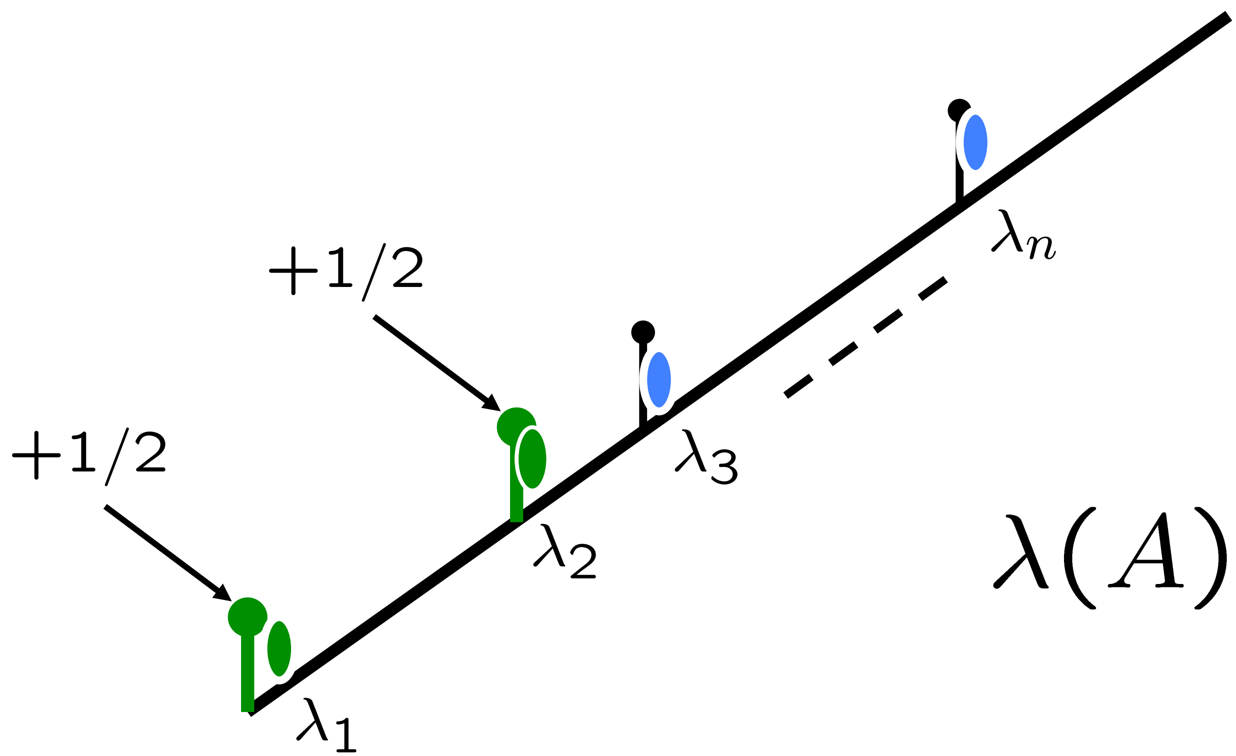


Ex2: Equal weight on u_1, u_2



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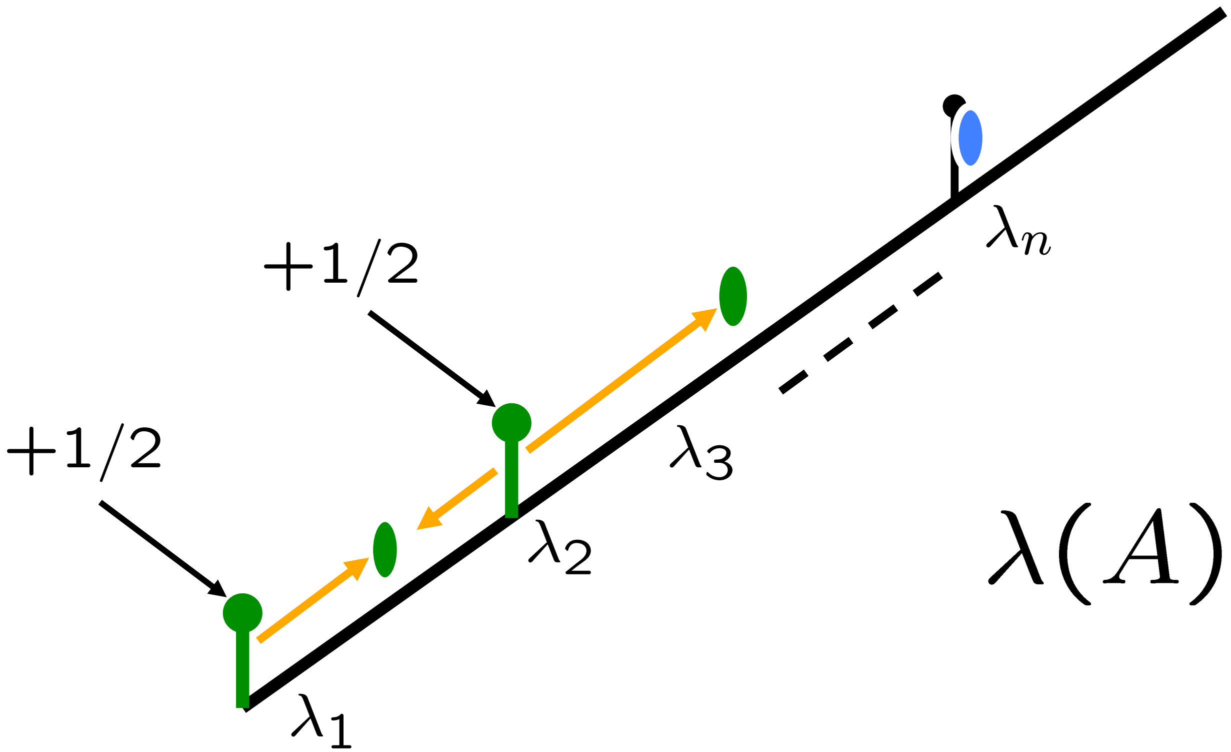

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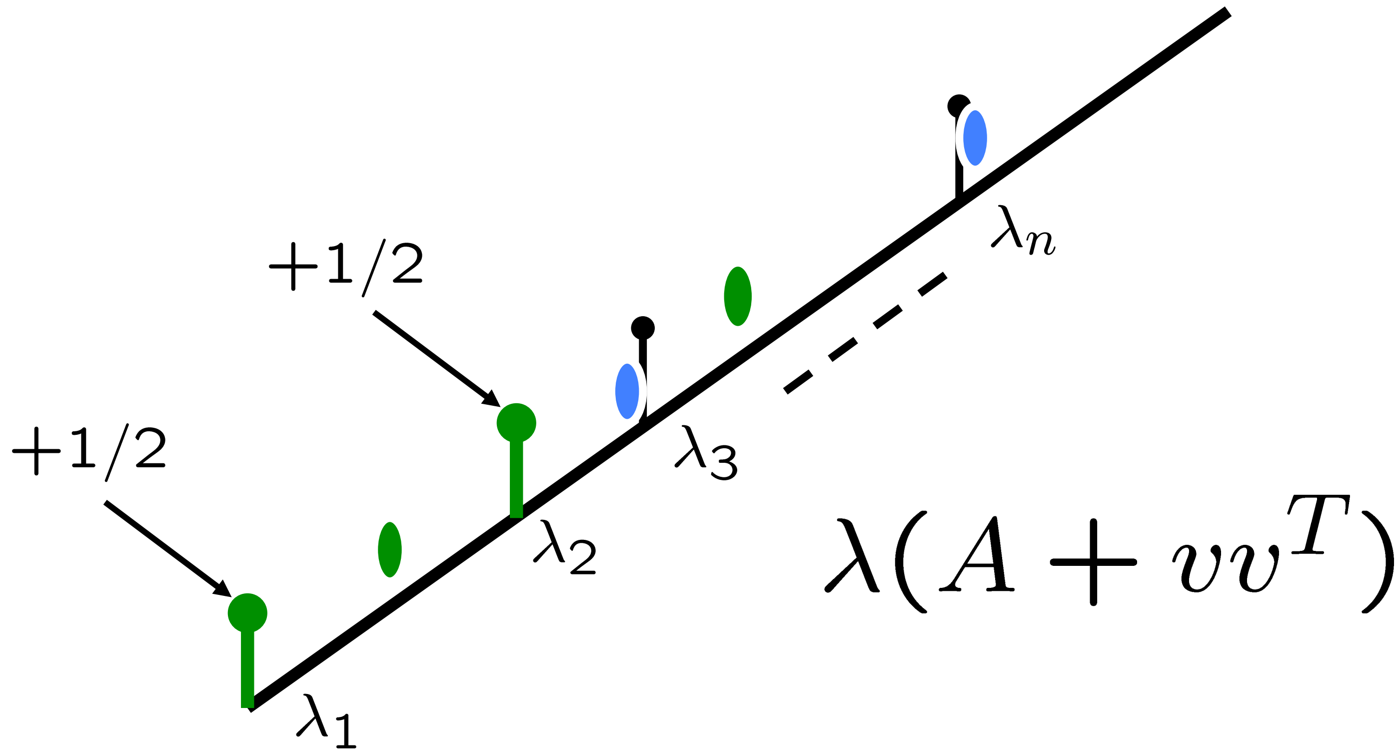
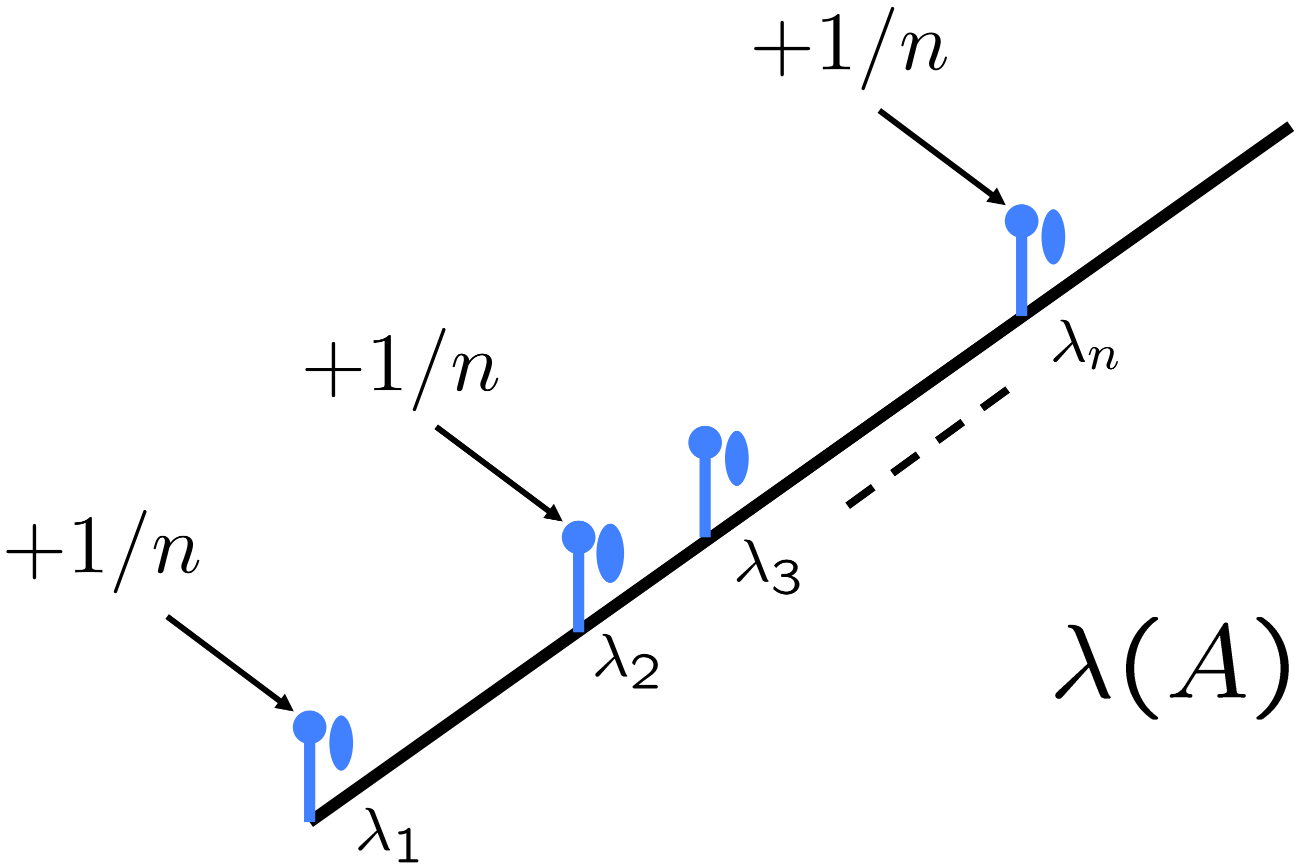


Ex3: Equal weight on all $u_1, u_2, \ldots, u_n$



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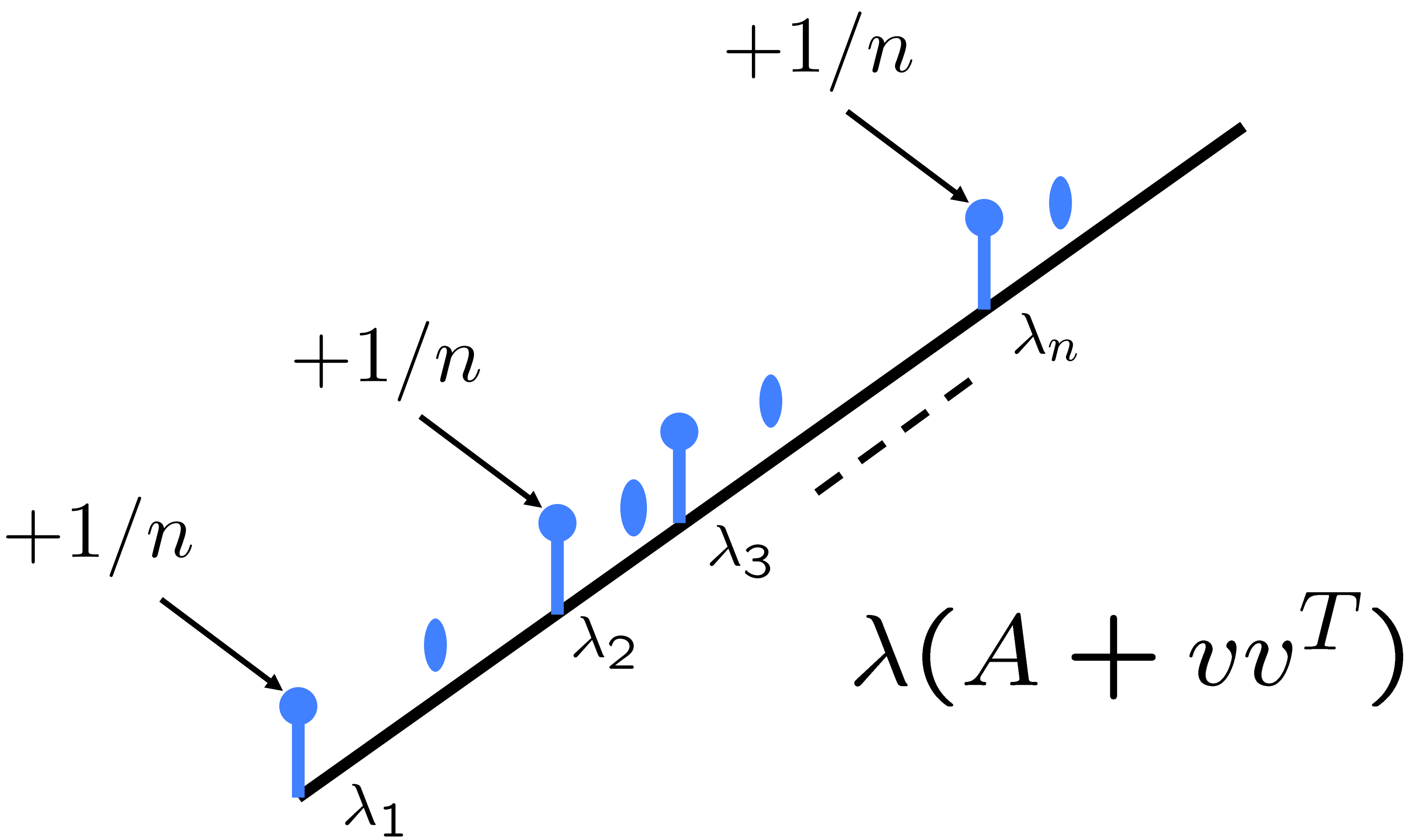


Adding a random v_e

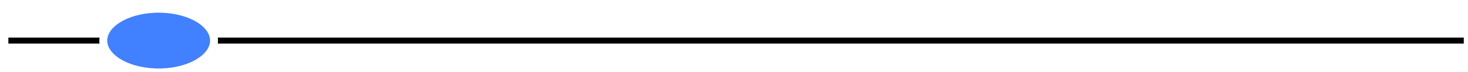
Because v_e are decomposition of identity,

$$\mathop{\mathbf{E}}_{e}\left[\ \langle v_e, u_i \rangle^2\ \right] = 1/m$$

$$\mathop{\mathbf{E}}_{e}\left[\ P_{A+v_e v_e^T}\ \right] = \left(1 + \frac{1}{m}\sum_i \frac{1}{\lambda_i - x}\right) P_A$$

$$= P_A - \frac{1}{m}\frac{d}{dx} P_A$$

Ideal proof

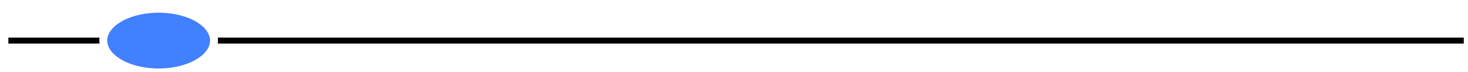


$$A^{(0)} = 0$$

$$p^{(0)} = x^n$$

Ideal proof

$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



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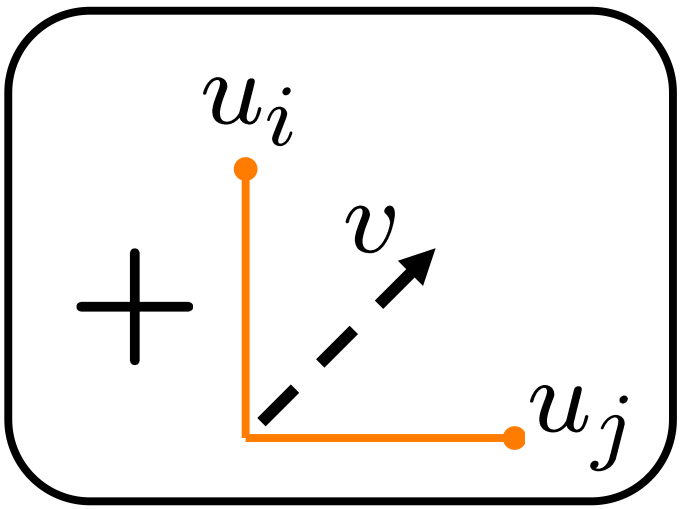
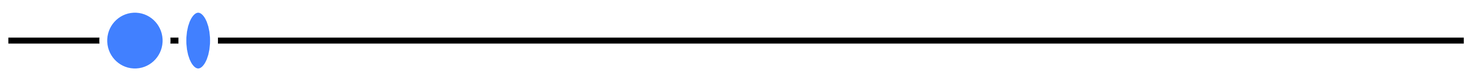


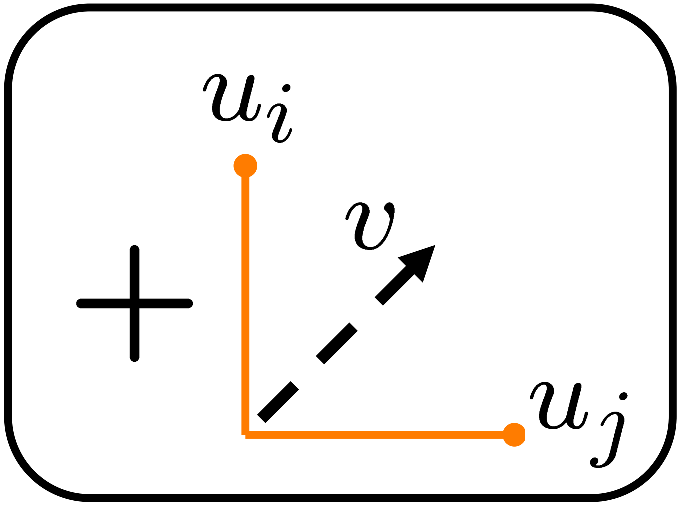
$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



$$A^{(1)} = 0 + vv^T$$

$$p^{(1)} = x^n - \frac{1}{m}\frac{d}{dx}x^n$$

Ideal proof

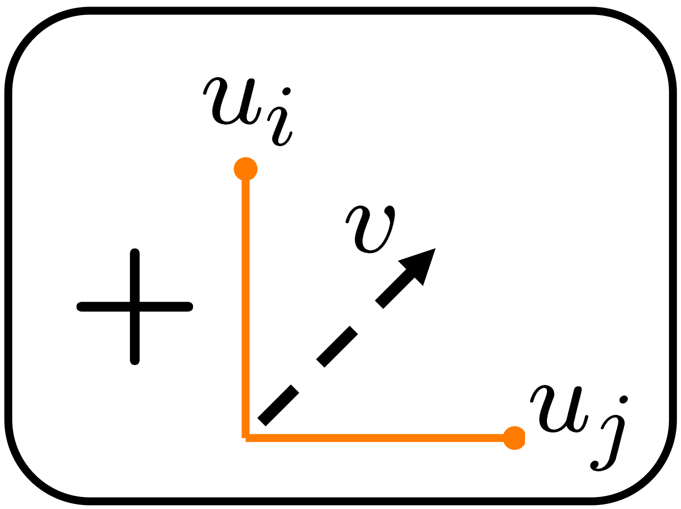


$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$

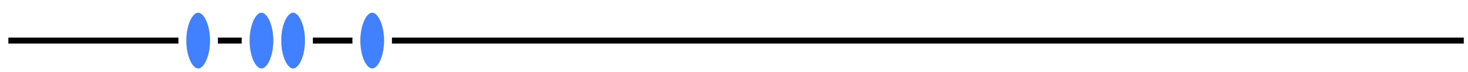
$$A^{(2)} = A^{(1)} + vv^T$$

$$p^{(2)} = \left(1 - \frac{1}{m}\frac{d}{dx}\right)^2 x^n$$

Ideal proof



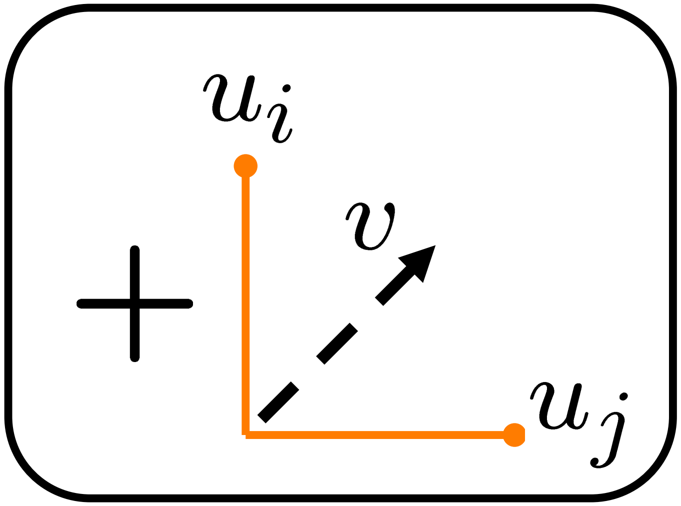
$$\mathbb{E}_e \langle v_e, u_i \rangle^2 = 1/m$$



$$A^{(i+1)} = A^{(i)} + vv^T$$

$$p^{(i+1)} = \left(1 - \frac{1}{m}\frac{d}{dx}\right)^{i+1} x^n$$

Ideal proof

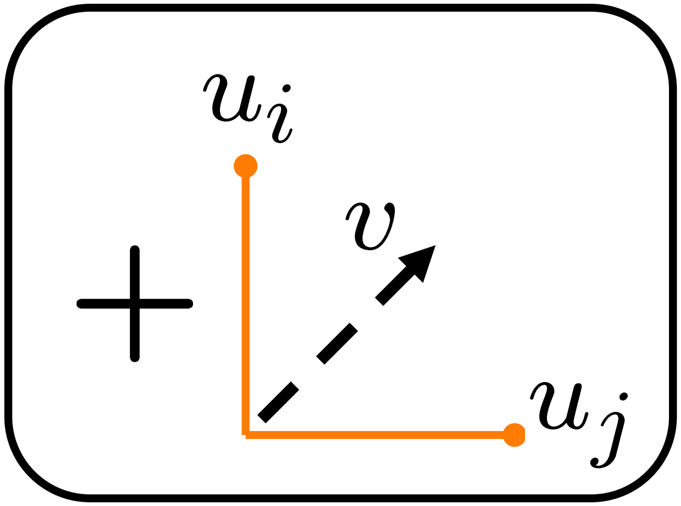


$$\mathbb{E}_e\langle v_e, u_i\rangle^2 = 1/m$$

$$A^{(i+1)} = A^{(i)} + vv^T$$

$$p^{(i+1)} = \left(1 - \frac{1}{m}\frac{d}{dx}\right)^{i+1} x^n$$

Ideal proof



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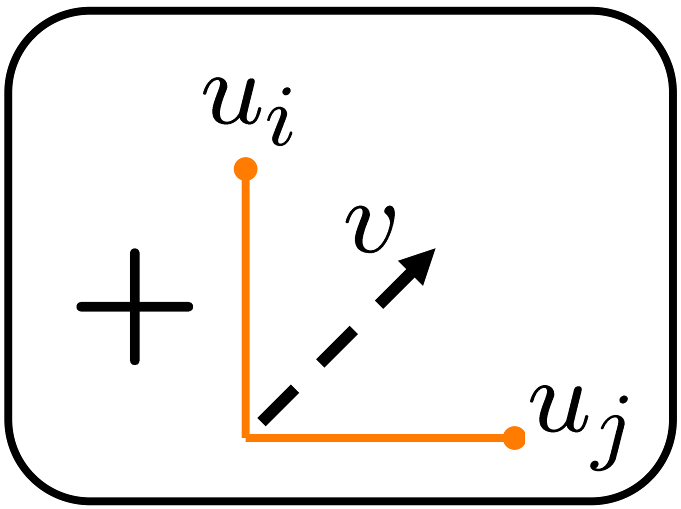

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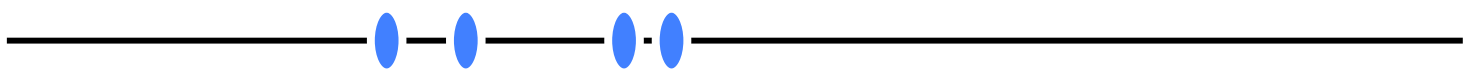

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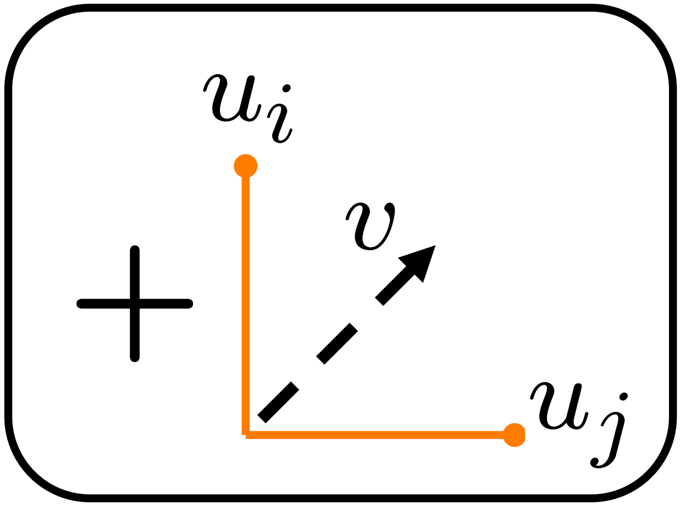

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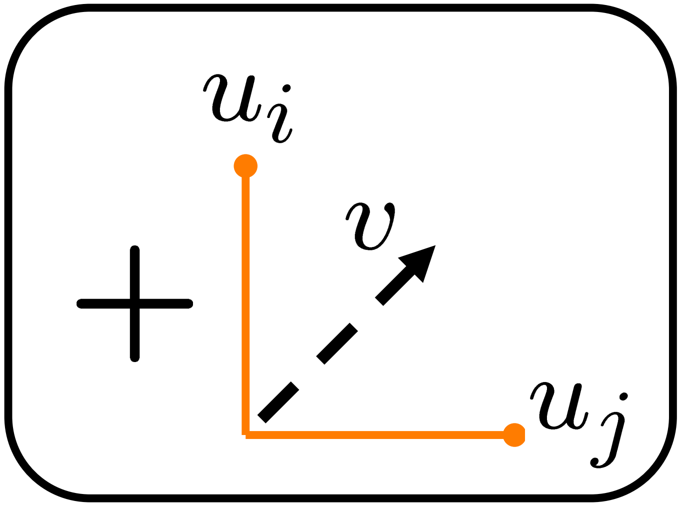

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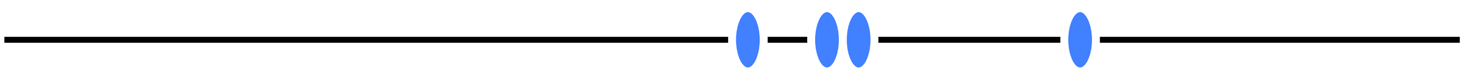

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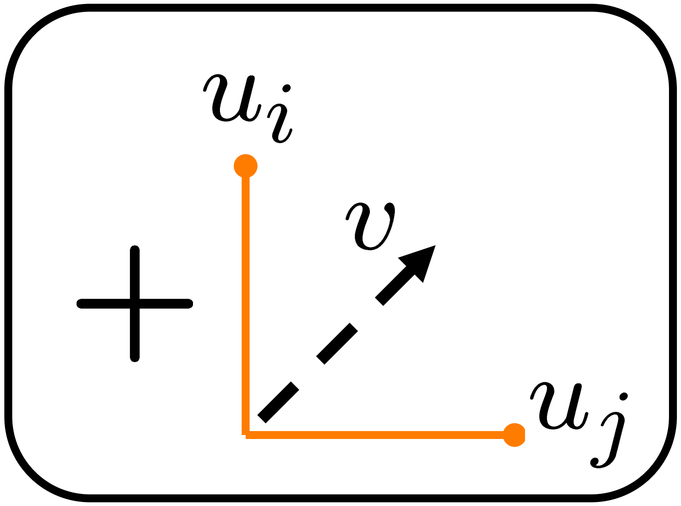

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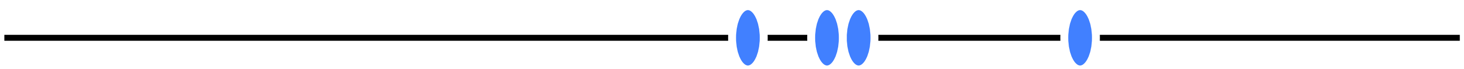

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....

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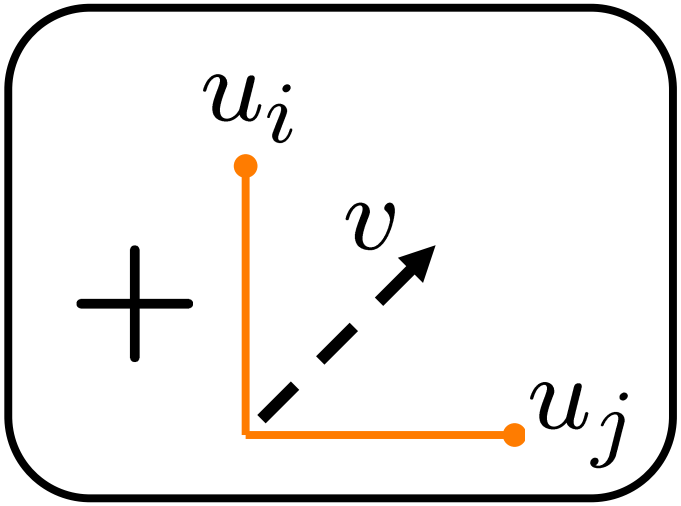
Ideal proof



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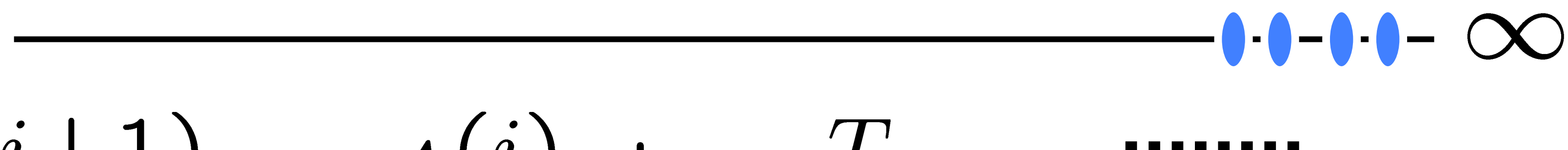

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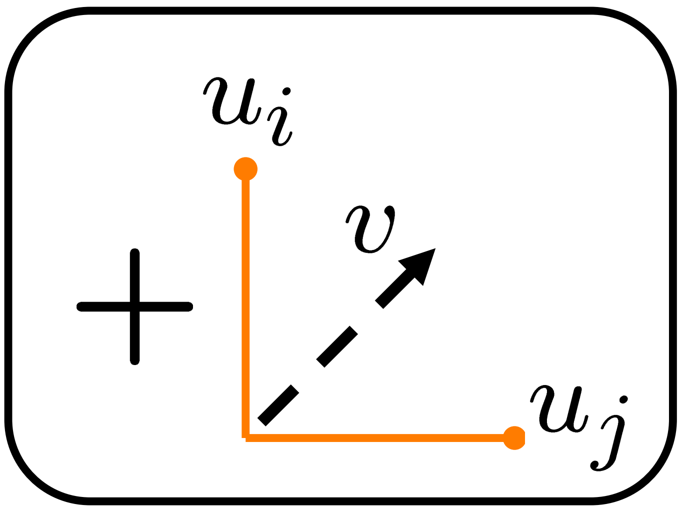

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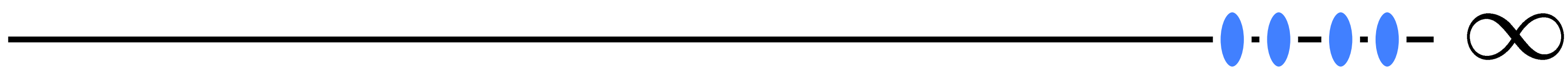
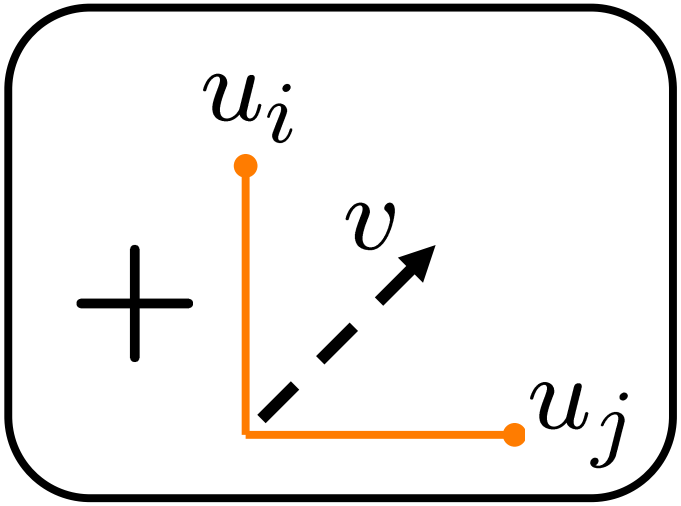


$$A^{(i+1)} = A^{(i)} + vv^T$$

$$p^{(i+1)} = \left(1 - \frac{1}{m}\frac{d}{dx}\right)^{i+1} x^n$$

$$\frac{\lambda_n(A)}{\lambda_1(A)} \leq 13?$$

Ideal proof



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∞

$$A^{(i+1)} = A^{(i)} + vv^T$$

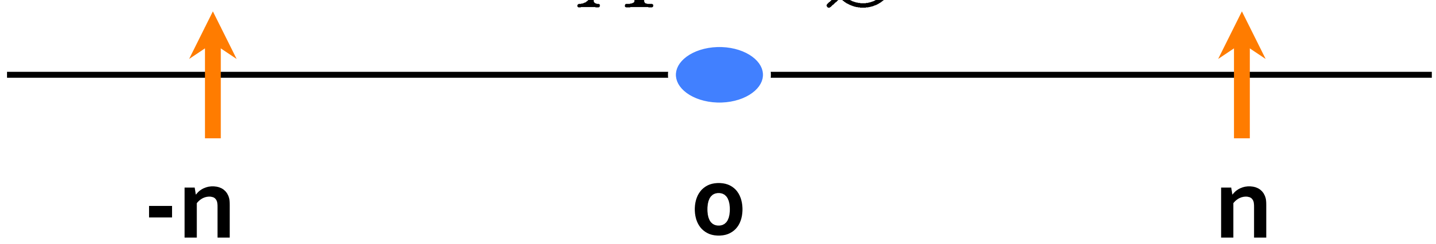
$$p^{(i+1)} = \left(1 - \frac{1}{m}\frac{d}{dx}\right)^{i+1} x^n$$

$$\frac{\lambda_n(A)}{\lambda_1(A)} \to 1$$

= associated Laguerre polynomial

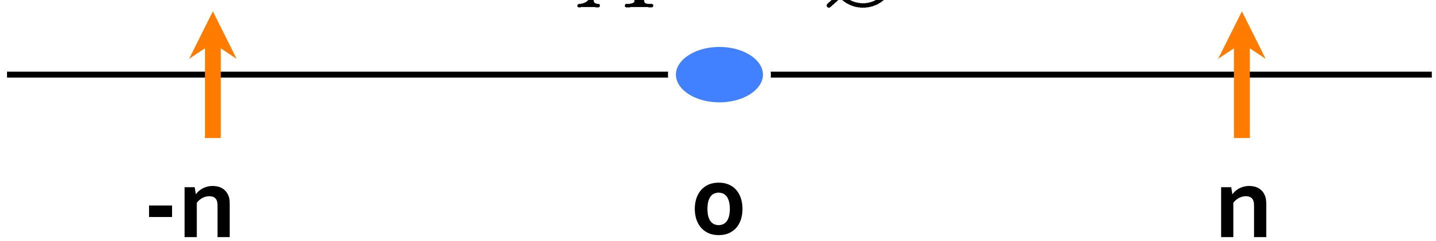
Broad outline: moving barriers

$$A = \varnothing$$



Step 1

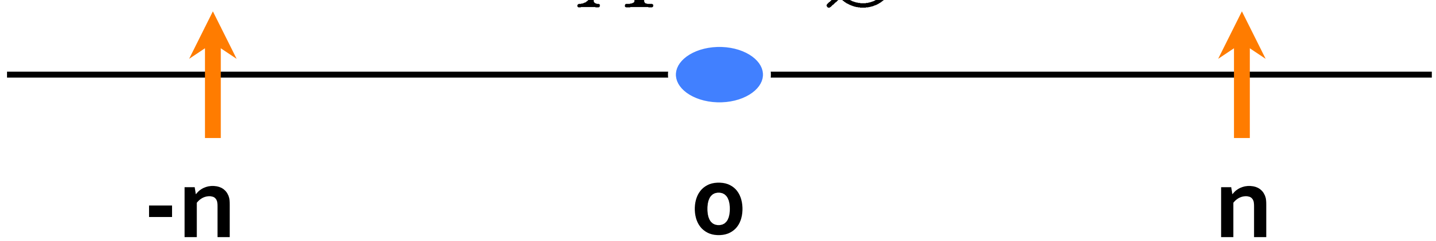
$$A = \varnothing$$



$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

Step 1

$$A = \varnothing$$



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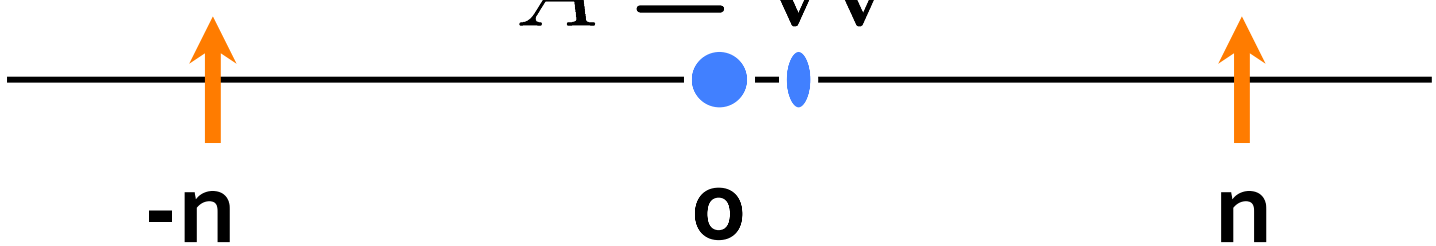
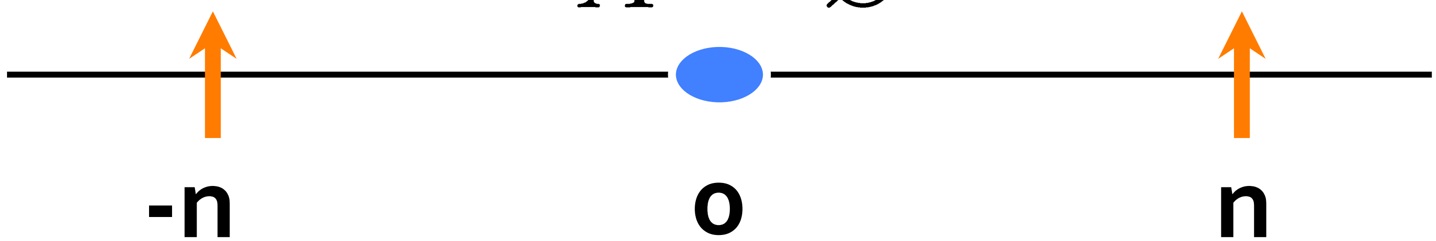
$$A = \mathbf{v}\mathbf{v}^T$$



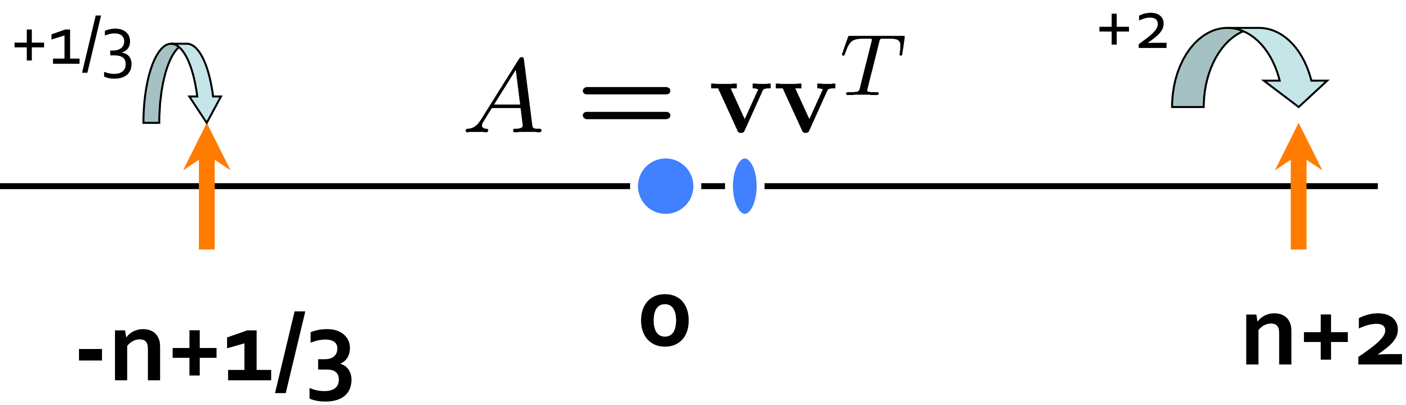
Step 1

$$A = \emptyset$$

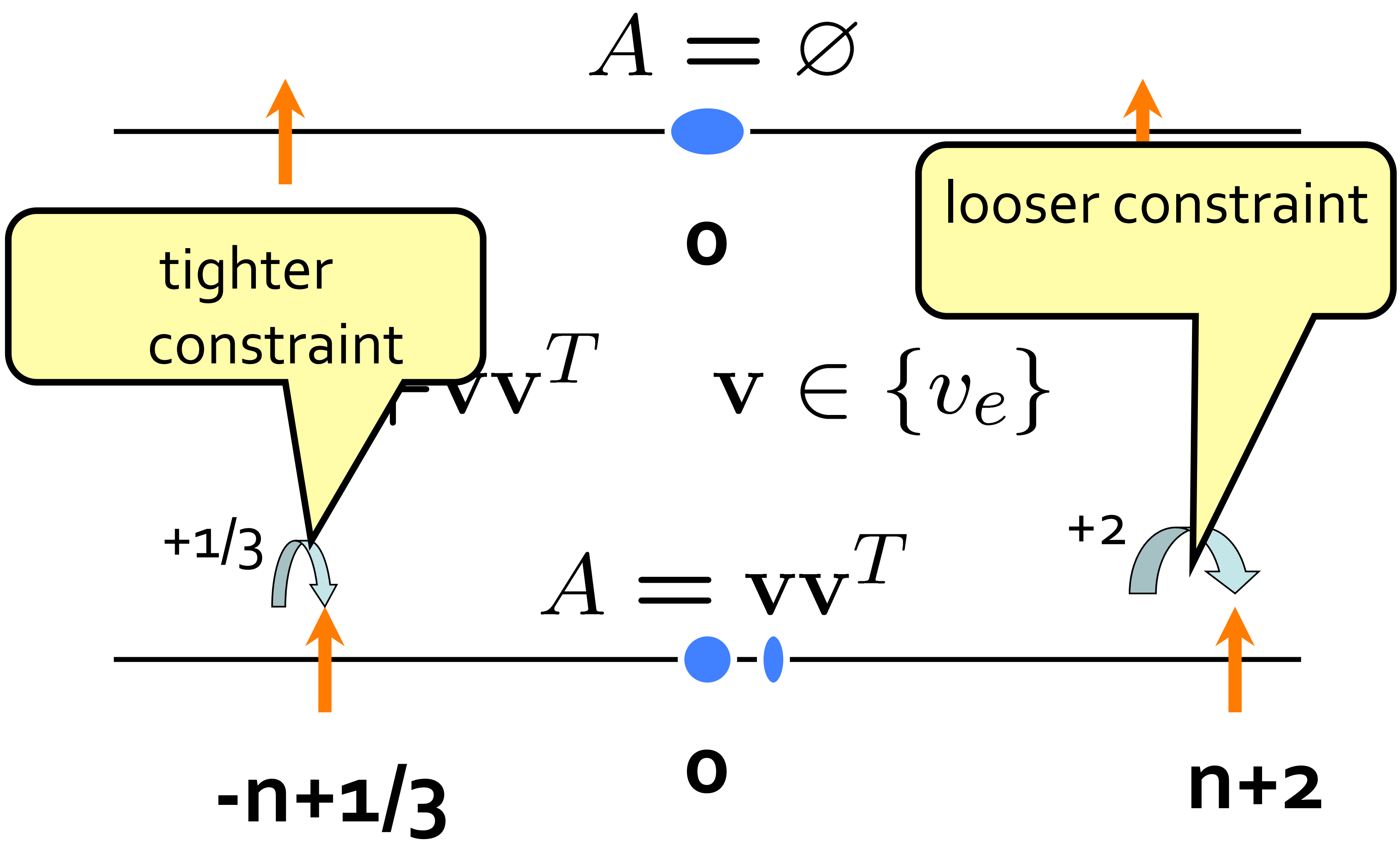


$$+\mathbf{v}\mathbf{v}^T \quad \mathbf{v} \in \{v_e\}$$

$$A = \mathbf{v}\mathbf{v}^T$$

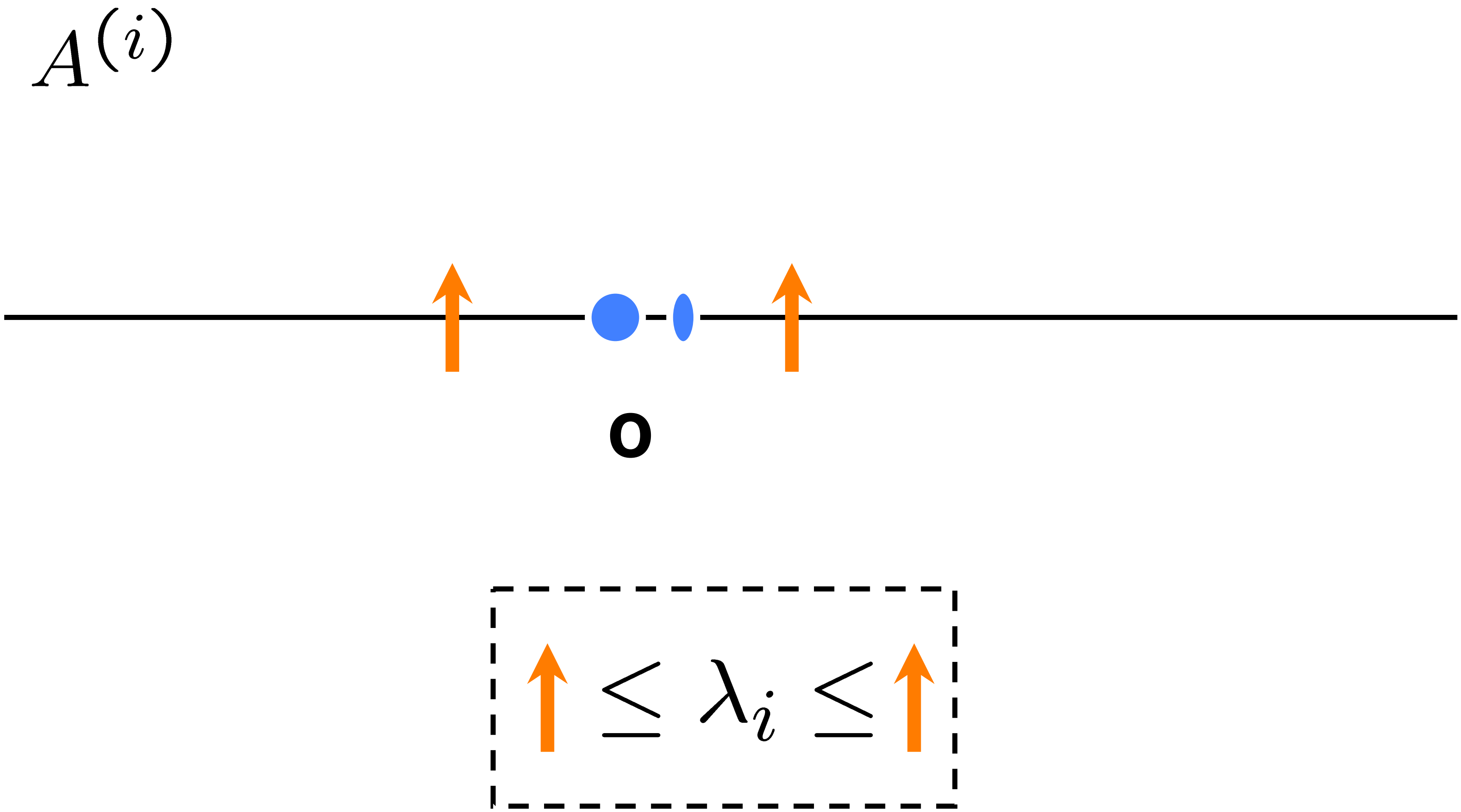


Step 1

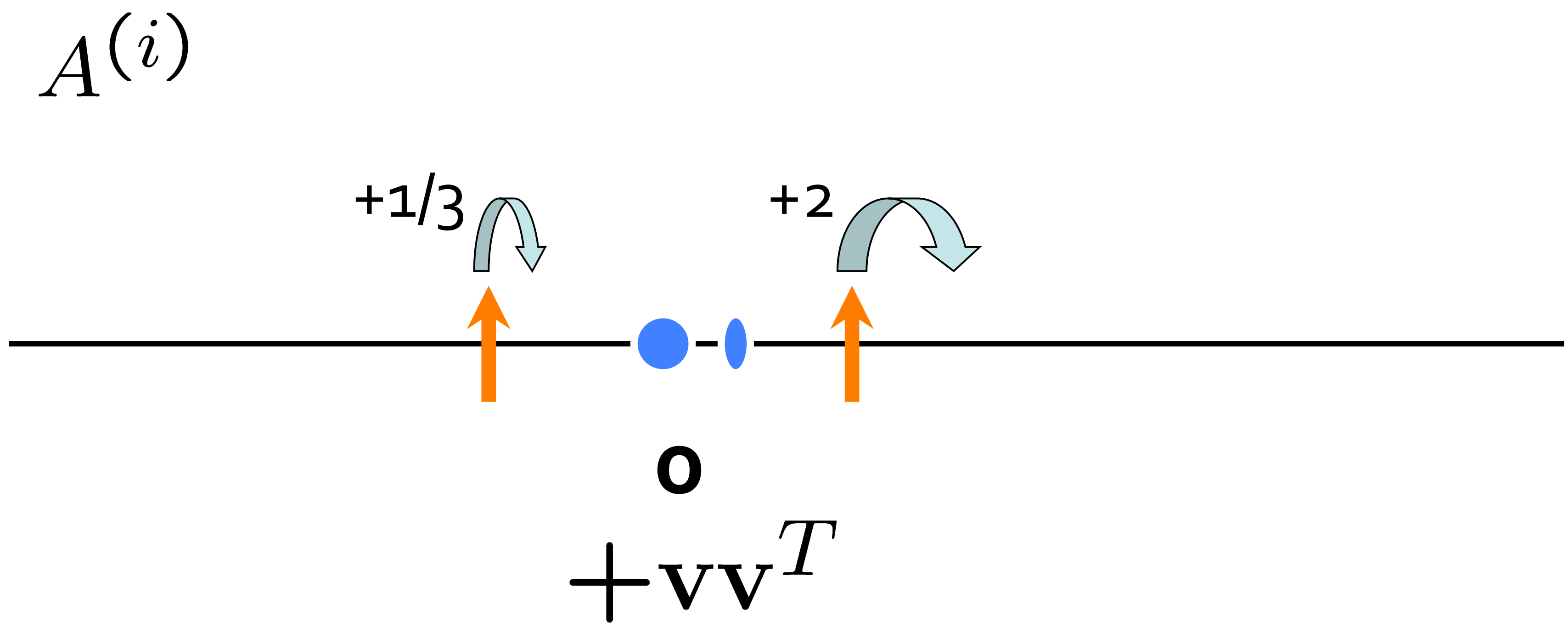


$A = \varnothing$
0
tighter constraint
looser constraint
$\mathbf{v}\mathbf{v}^T$
$\mathbf{v} \in \{v_e\}$
+1/3
+2
$A = \mathbf{v}\mathbf{v}^T$
0
-n+1/3
n+2

Step i+1

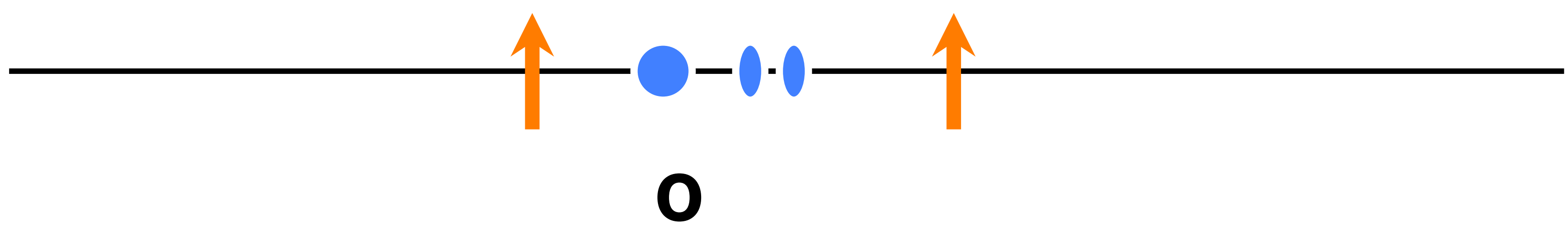


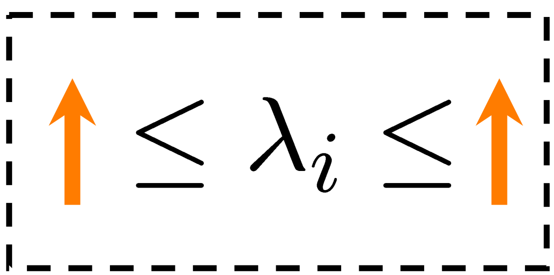
Step i+1



Step i+1

$$A^{(i)}, A^{(i+1)}$$





Step i+1

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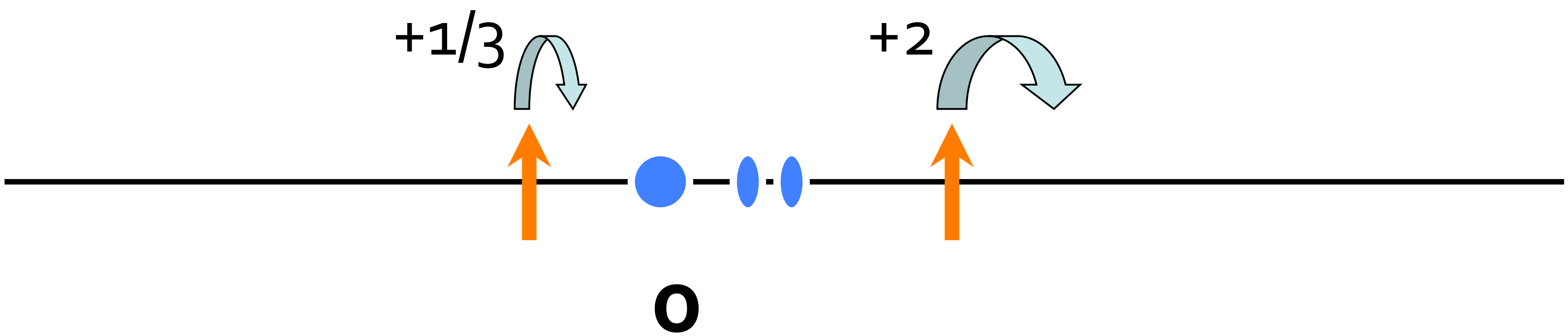
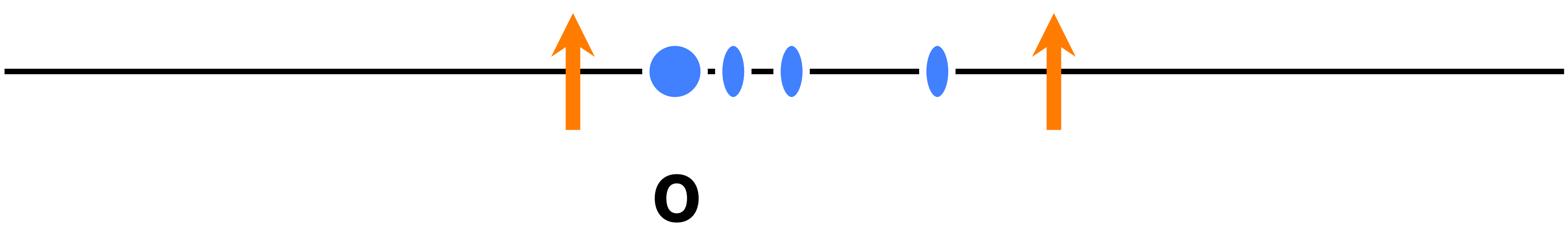
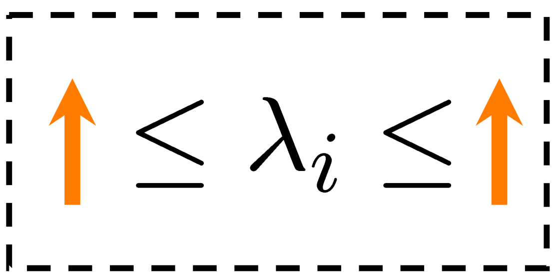


$+\mathbf{v}\mathbf{v}^T$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$





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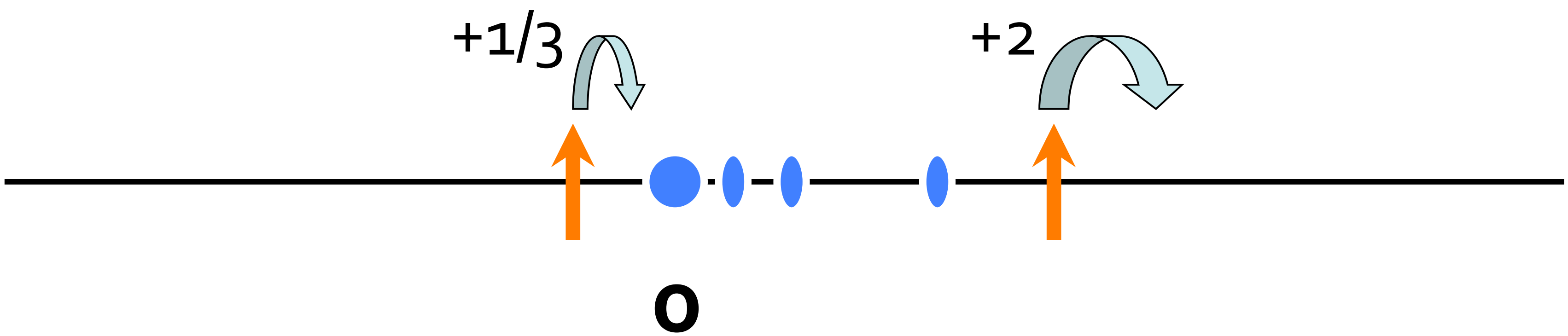
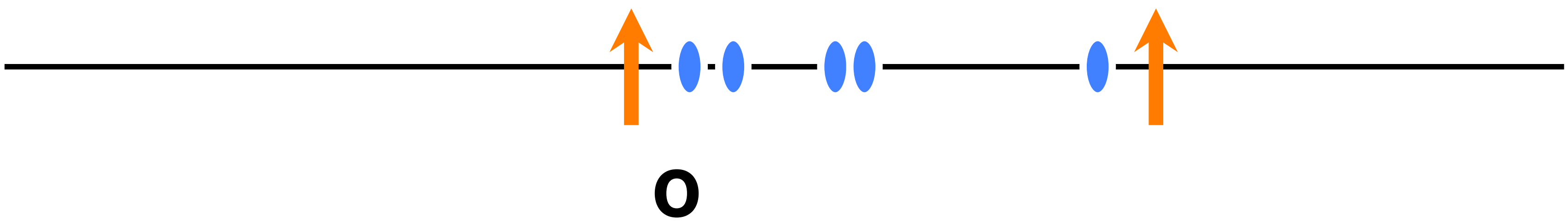
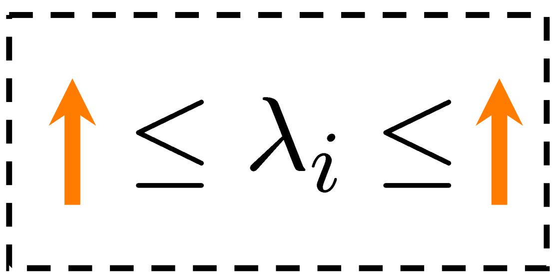


$$+\mathbf{v}\mathbf{v}^T$$

Step i+1

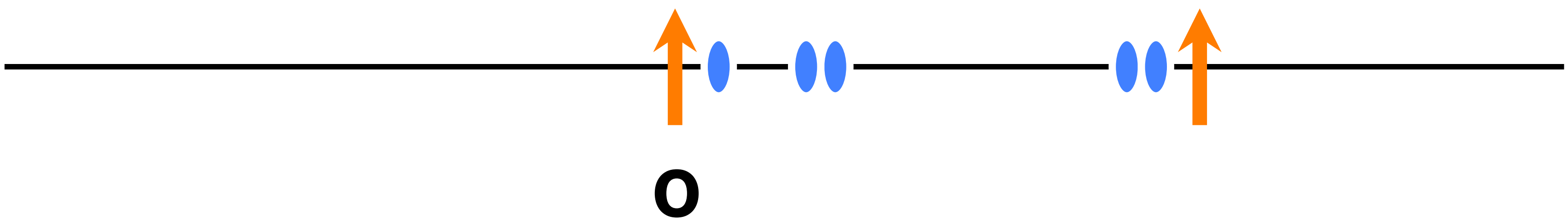
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$

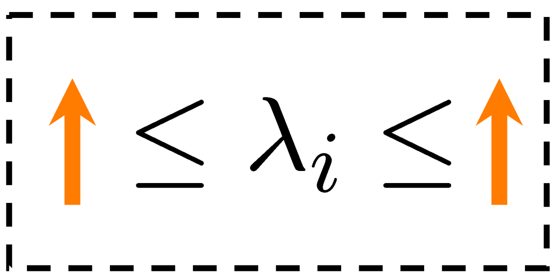




Step i+1

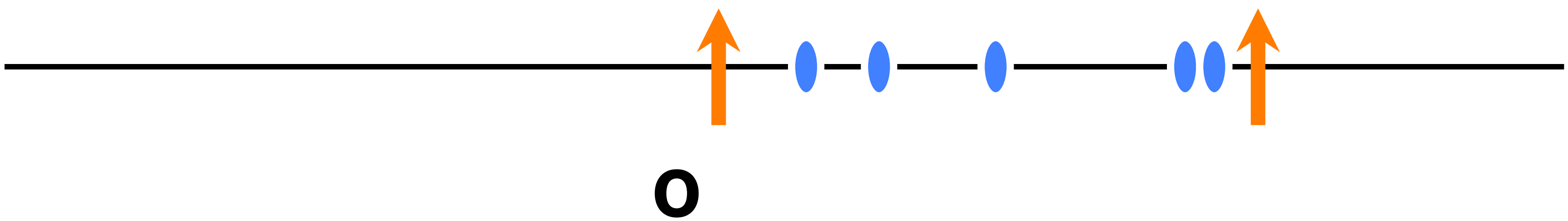
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

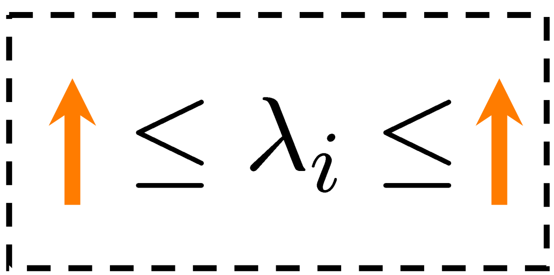




Step i+1

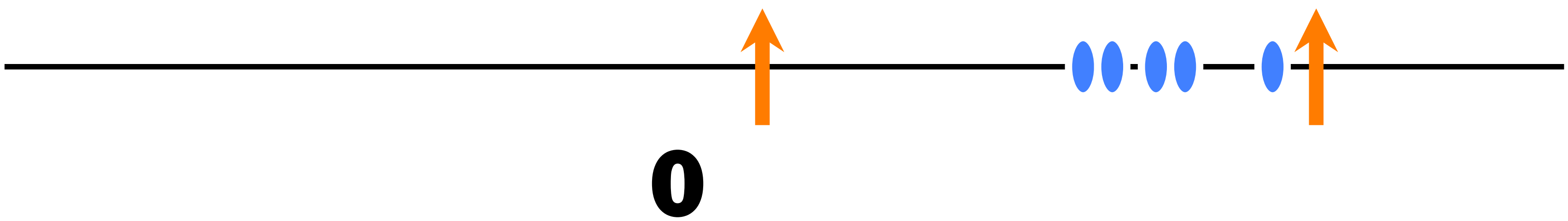
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$

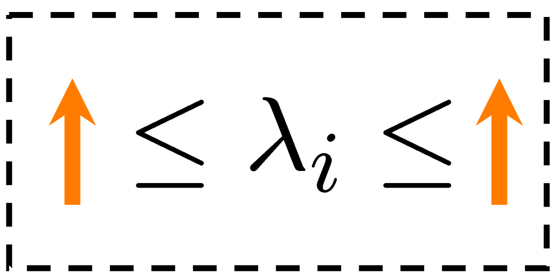




Step i+1

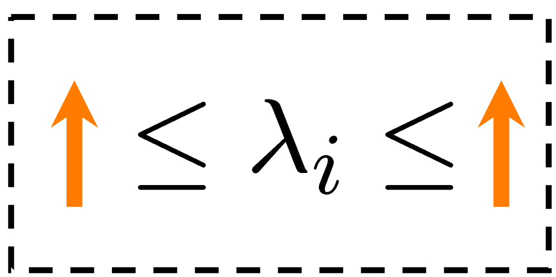
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$





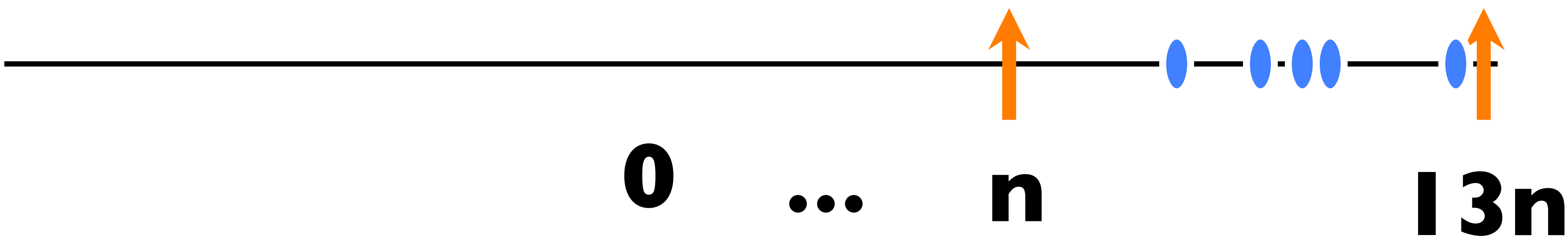
Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



2.6-approximation with 6n vectors.

□

Problem

need to show that an appropriate

$$v_e v_e^T$$

always exists.

Problem

need to show that an appropriate

$$v_e v_e^T$$

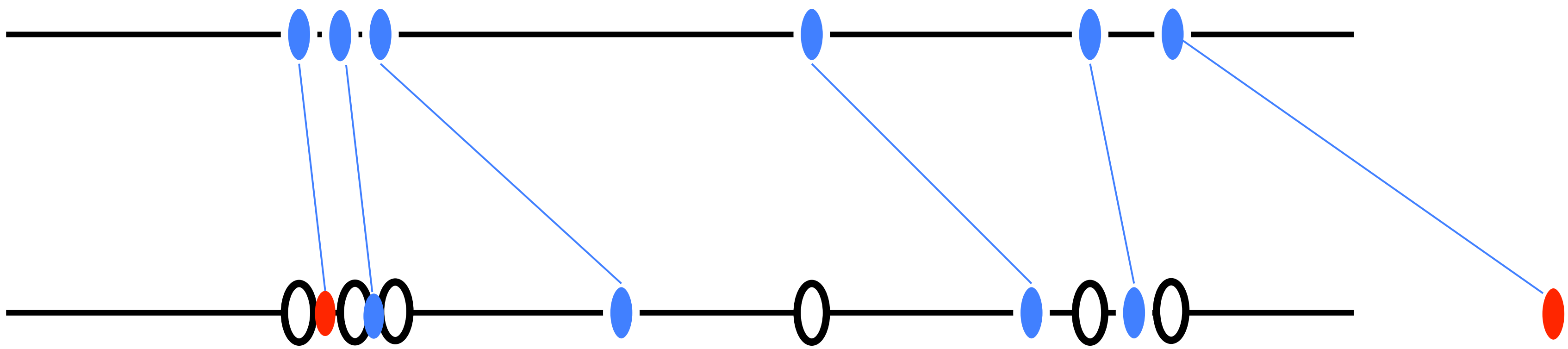
always exists.

$$\uparrow \leq \lambda_i \leq \uparrow$$

Is not strong enough for induction

Problems

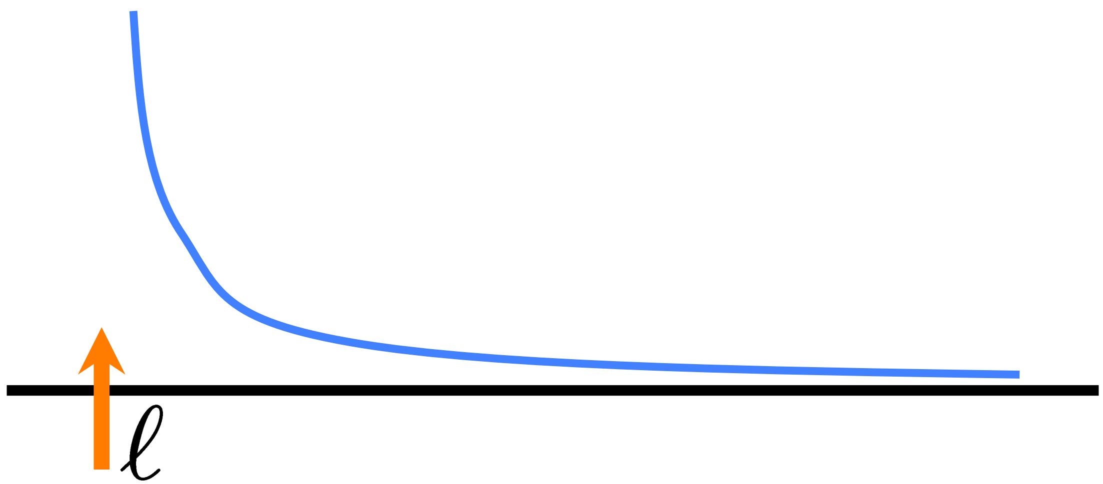
If many small eigs, can only move one



If v_e has large inner product with large eigenvectors, the largest eigenvalue moves far.

The Lower Barrier Potential Function

$$\Phi_\ell(A) = \sum_i \frac{1}{\lambda_i - \ell} = \mathrm{Tr}\left((A - \ell I)^{-1}\right)$$



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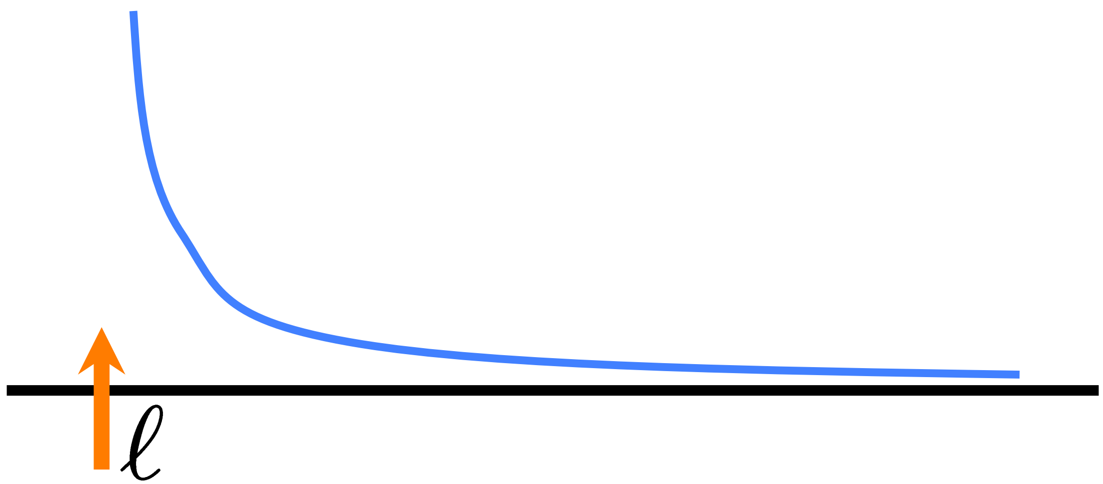

$$\Phi_\ell(A) \leq 1 \implies \lambda_{min}(A) \geq \ell + 1$$

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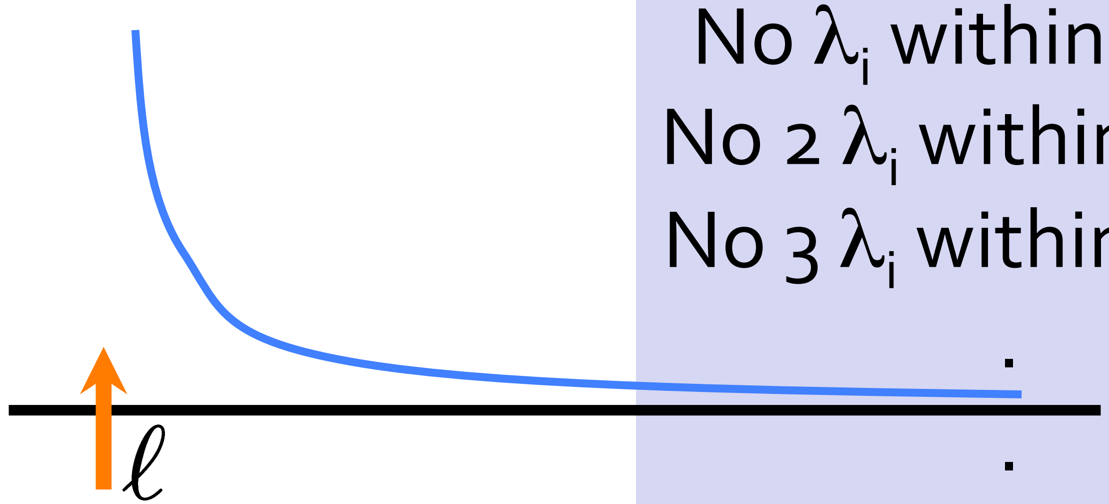
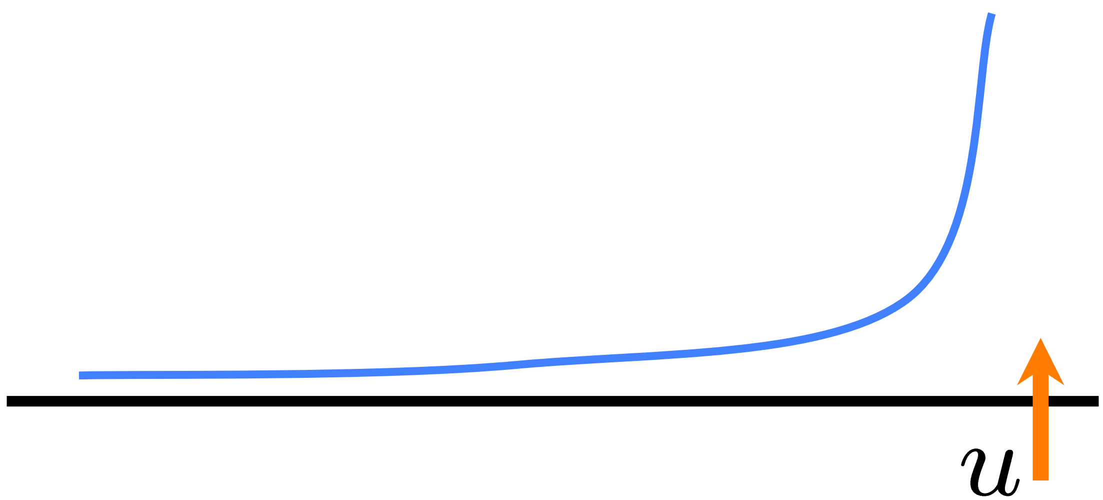


No λ_i within dist. 1
No 2 λ_i within dist. 2
No 3 λ_i within dist. 3
.
.
No k λ_i within dist. k

$$\Phi_\ell(A) \le 1 \implies \lambda_{min}(A) \ge \ell + 1$$

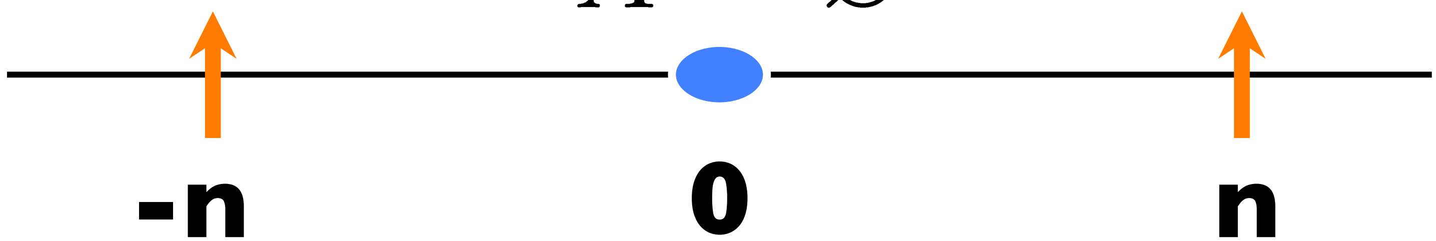
The Upper Barrier Potential Function

$$\Phi^u(A) = \sum_i \frac{1}{u-\lambda_i} = \mathrm{Tr}\left((uI - A)^{-1}\right)$$



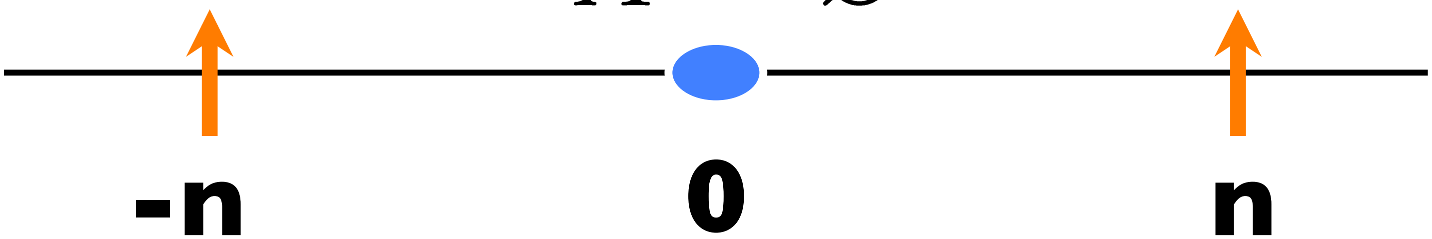
The Beginning

$$A = \varnothing$$



The Beginning

$$A = \varnothing$$

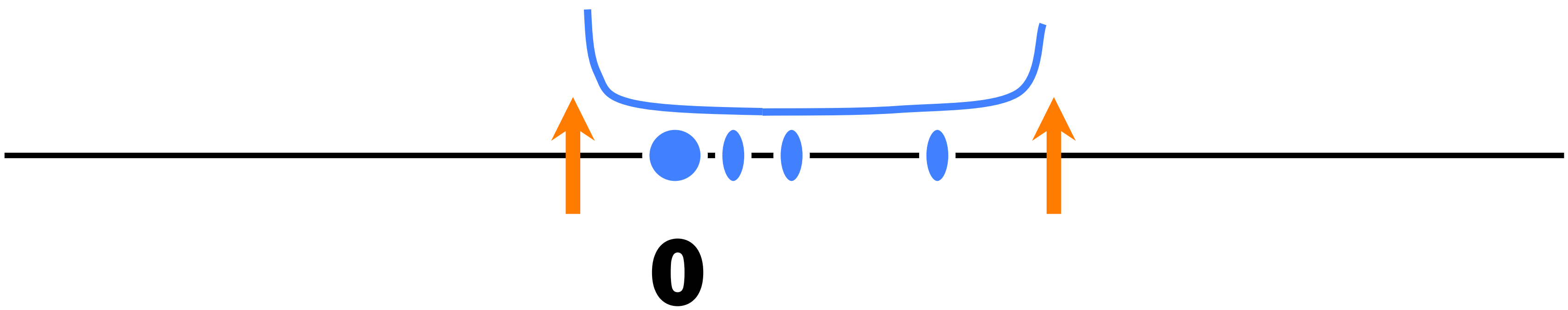


$$\Phi^{n}(\varnothing) = \mathrm{Tr}(nI)^{-1} = 1$$

$$\Phi_{-n}(\varnothing) = \mathrm{Tr}(nI)^{-1} = 1.$$

Step i+1

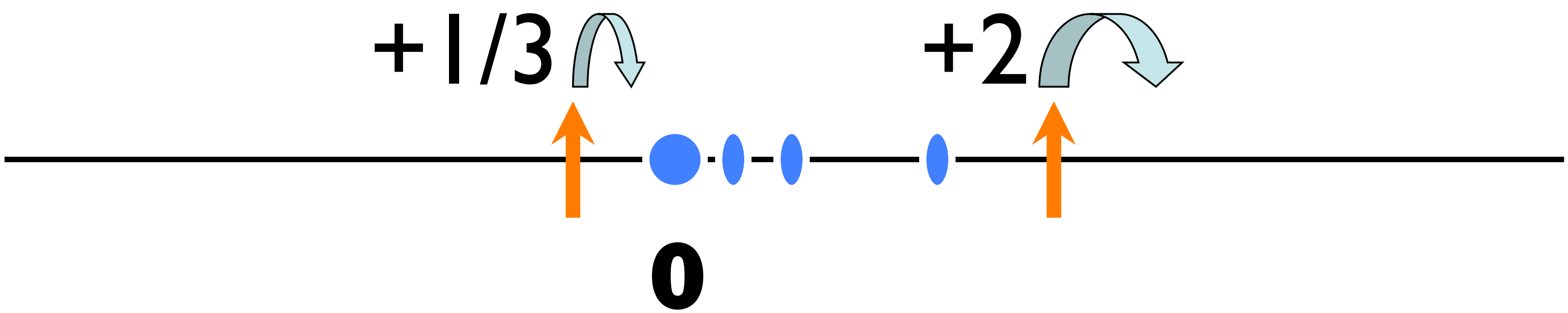
$A^{(i)}, A^{(i+1)}, A^{(i+2)}$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

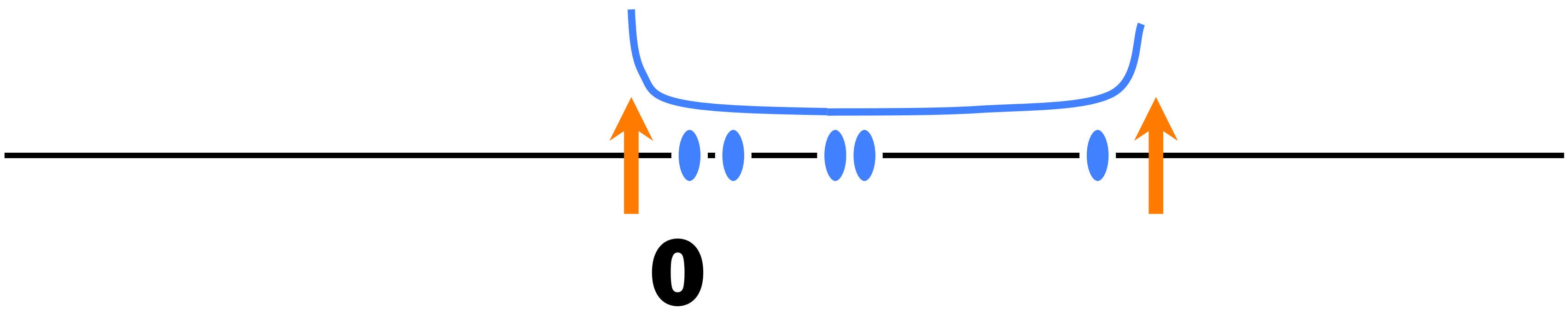
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$



Lemma.
can always choose $+s\mathbf{v}\mathbf{v}^T$ $\Phi^u(A) \leq 1$
so that potentials do not increase $\Phi_\ell(A) \leq 1.$

Step i+1

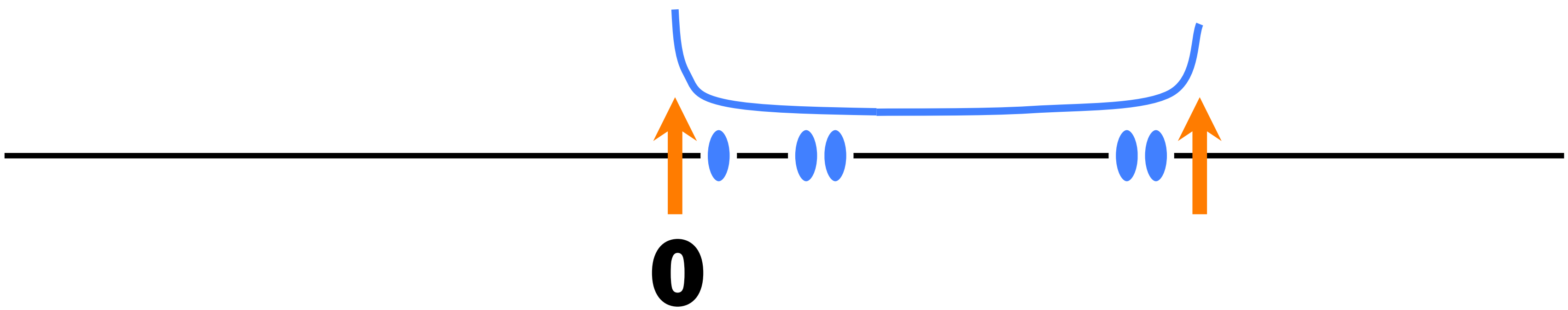
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

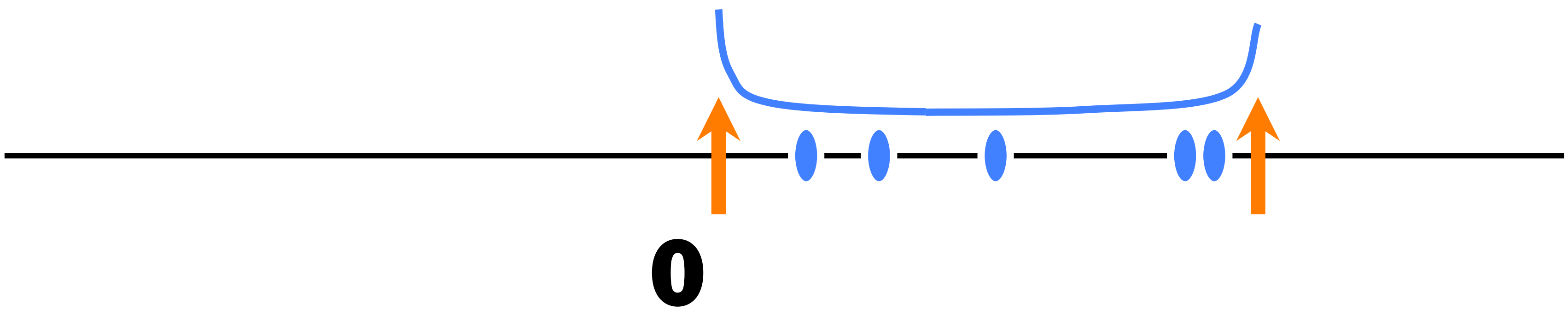
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

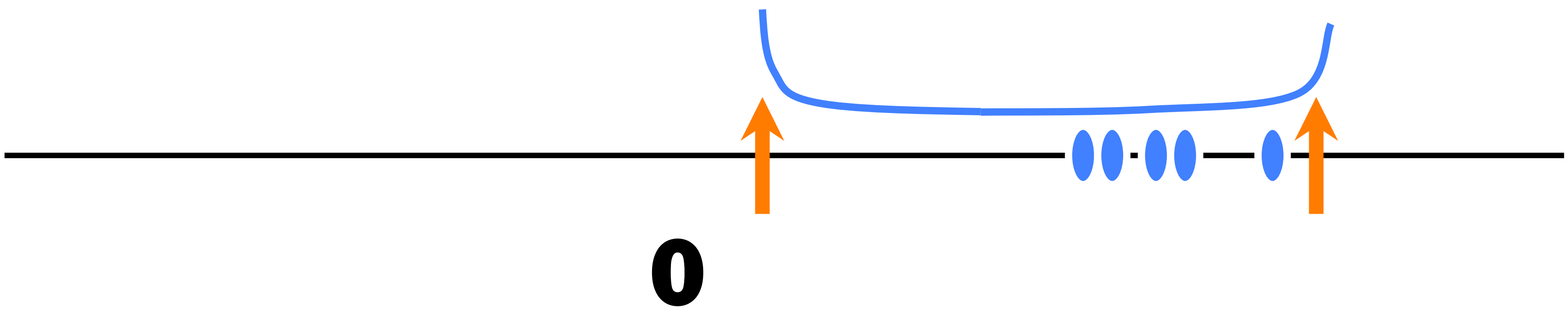
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$

$$\Phi^u(A) \leq 1$$

$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$

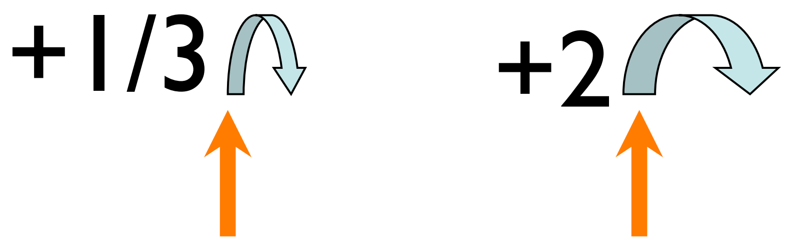
2.6-approximation with 6n vectors.

□

Goal

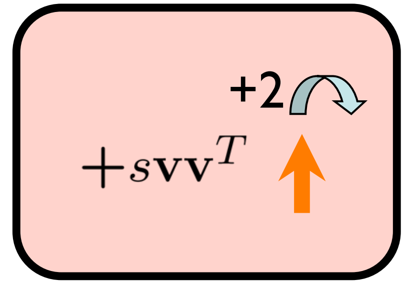
Lemma.
can always choose $+s\mathbf{v}\mathbf{v}^T$ so that potentials do not increase

$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$



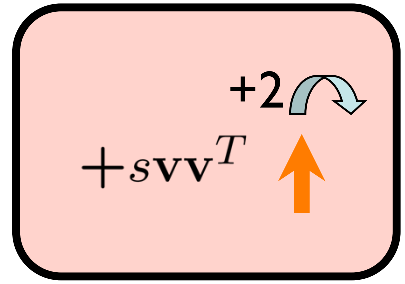
$$+s\mathbf{v}\mathbf{v}^T$$

Upper Barrier Update



Add $s\boldsymbol{v}\boldsymbol{v}^T$ and set $u' \leftarrow u + 2$.

Upper Barrier Update



Add $s\boldsymbol{v}\boldsymbol{v}^T$ and set $u' \leftarrow u + 2$.

$$\Phi^{u'}(A + s\boldsymbol{v}\boldsymbol{v}^T)$$

$$= \mathrm{Tr}\left((u'I - A - s\boldsymbol{v}\boldsymbol{v}^T)^{-1}\right)$$

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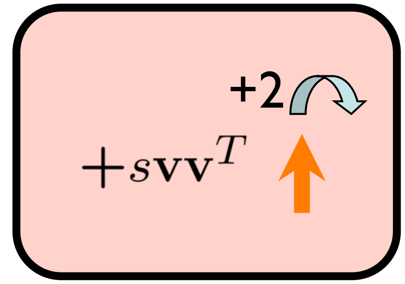

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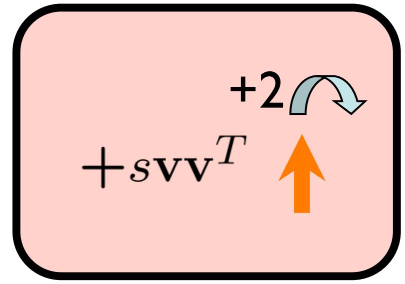
$$= \mathrm{Tr}\left((u'I - A - s\boldsymbol{v}\boldsymbol{v}^T)^{-1}\right)$$

$$= \Phi^{u'}(A) + \frac{s\boldsymbol{v}^T(u'I - A)^{-2}\boldsymbol{v}}{1 - s\boldsymbol{v}^T(u'I - A)^{-1}\boldsymbol{v}}$$

By Sherman-Morrison Formula

Upper Barrier Update



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Need $\leq \Phi^{u}(A)$

How much of $\mathbf{v}\mathbf{v}^T$ can we add?

Rearranging:

$$\Phi^{u'}(A + s\boldsymbol{v}\boldsymbol{v}^T) \leq \Phi^{u}(A)$$

iff

$$1 \geq s\boldsymbol{v}^T \left(\frac{(u'I - A)^{-2}}{\Phi^{u}(A) - \Phi^{u'}(A)} + (u'I - A)^{-1} \right) \boldsymbol{v}$$

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Write as $1 \geq s\boldsymbol{v}^T U_A \boldsymbol{v}$

Lower Barrier

Similarly:

$$\Phi_{l'}(A + s\boldsymbol{v}\boldsymbol{v}^T) \leq \Phi_l(A)$$

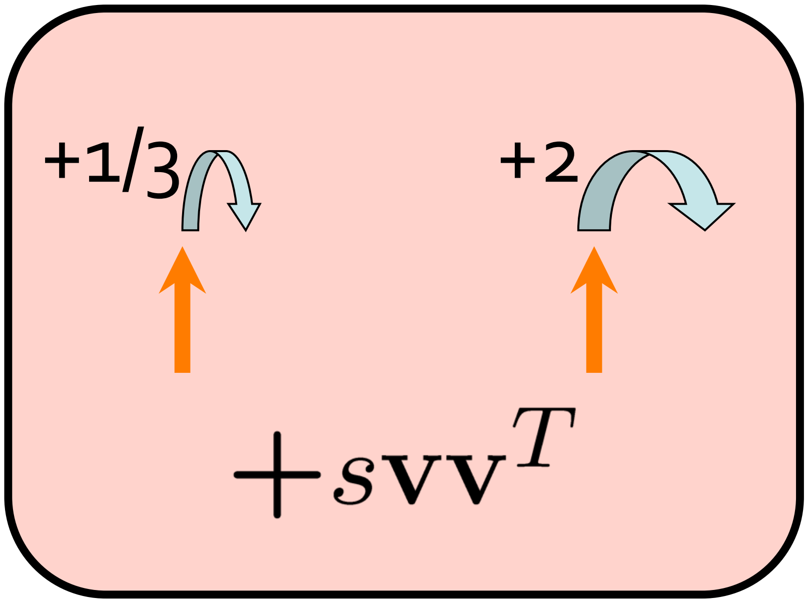
iff

$$1 \leq s\boldsymbol{v}^T \left(\frac{(A - l'I)^{-2}}{\Phi_{l'}(A) - \Phi_l(A)} - (A - l'I)^{-1} \right) \boldsymbol{v}$$

Write as $$1 \leq s\boldsymbol{v}^T L_A \boldsymbol{v}$$

Goal

Show that we can always add some vector while respecting *both* barriers.



Need: $$s\boldsymbol{v}^T U_A \boldsymbol{v} \leq 1 \leq s\boldsymbol{v}^T L_A \boldsymbol{v}$$

Two expectations

Need: $$s\boldsymbol{v}^T U_A \boldsymbol{v} \leq 1 \leq s\boldsymbol{v}^T L_A \boldsymbol{v}$$

Can show: $$\mathbf{E}_e \left[\, \boldsymbol{v}_e^T U_A \boldsymbol{v}_e \, \right] \leq 1$$

$$\mathbf{E}_e \left[\, \boldsymbol{v}_e^T L_A \boldsymbol{v}_e \, \right] \geq 1$$

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So: $\mathbf{E}_e \left[\; \boldsymbol{v}_e^T U_A \boldsymbol{v}_e \; \right] \leq \mathbf{E}_e \left[\; \boldsymbol{v}_e^T L_A \boldsymbol{v}_e \; \right]$

And, exists $e : \boldsymbol{v}_e^T U_A \boldsymbol{v}_e \leq \boldsymbol{v}_e^T L_A \boldsymbol{v}_e$

Two expectations

Need: $s \boldsymbol{v}^T U_A \boldsymbol{v} \leq 1 \leq s \boldsymbol{v}^T L_A \boldsymbol{v}$

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And, exists $e : \boldsymbol{v}_e^T U_A \boldsymbol{v}_e \leq \boldsymbol{v}_e^T L_A \boldsymbol{v}_e$

And s that puts 1 between them

Bounding expectations

$$v^T U_A v = \mathrm{Tr}\left(U_A v v^T\right)$$

$$\begin{aligned} \mathop{\mathbf{E}}_{e}\left[\ \mathrm{Tr}\left(U_A v_e v_e^T\right)\ \right] &= \mathrm{Tr}\left(U_A \mathop{\mathbf{E}}_{e}\left[\ v_e v_e^T\ \right]\right) \\ &= \mathrm{Tr}\left(U_A I\right) \\ &= \mathrm{Tr}\left(U_A\right) \end{aligned}$$

Bounding expectations

$$\mathrm{Tr}\left(U_A\right) = \frac{\mathrm{Tr}\left(\left(u'I - A\right)^{-2}\right)}{\Phi^{u}(A) - \Phi^{u'}(A)} + \mathrm{Tr}\left(\left(u'I - A\right)^{-1}\right)$$

$$\leq \frac{1}{u' - u}$$

$$= \Phi^{u'}(A)$$

$$\leq \frac{1}{2} + 1 = \frac{3}{2}$$

Bounding expectations

Similarly,

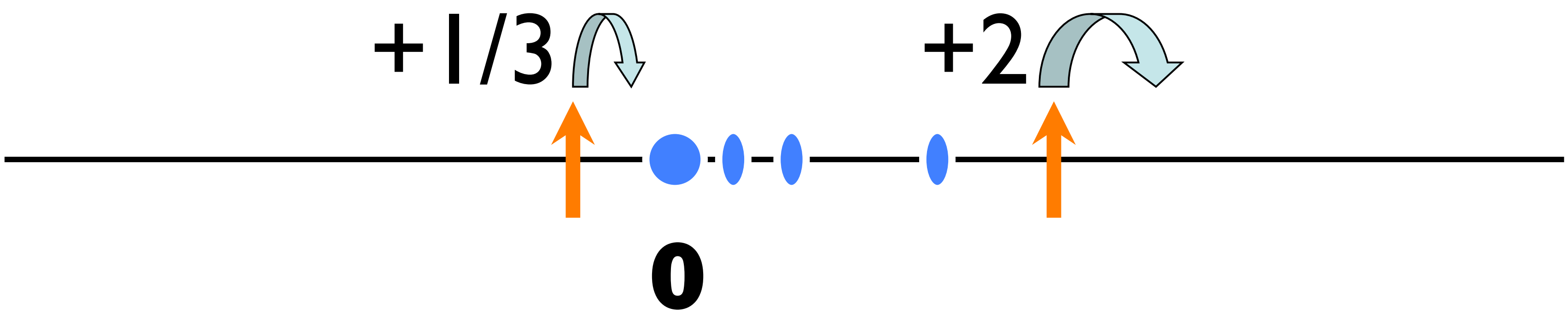
$$\mathrm{Tr}\left(L_A\right) \geq \frac{1}{l' - l} - 1$$
$$= \frac{1}{1/3} - 1$$
$$= 2$$

So

$$\mathbf{E}_e \left[\ \boldsymbol{v}_e^T U_A \boldsymbol{v}_e\ \right] = \mathrm{Tr}\left(U_A\right) \leq \mathrm{Tr}\left(L_A\right) = \mathbf{E}_e \left[\ \boldsymbol{v}_e^T L_A \boldsymbol{v}_e\ \right]$$

Step i+1

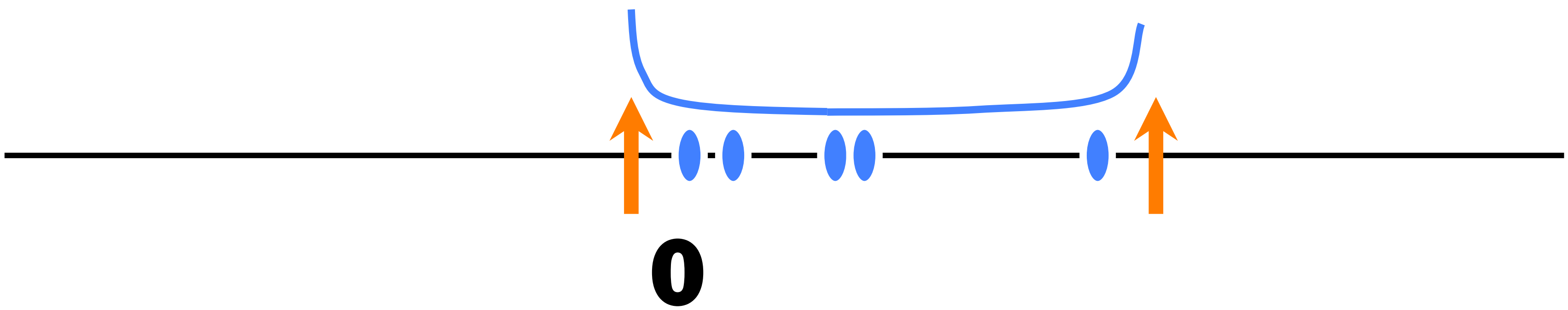
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}$$



Lemma.
can always choose $+s\mathbf{v}\mathbf{v}^T$
so that potentials do not increase $\Phi^u(A) \leq 1$ $\Phi_\ell(A) \leq 1.$

Step i+1

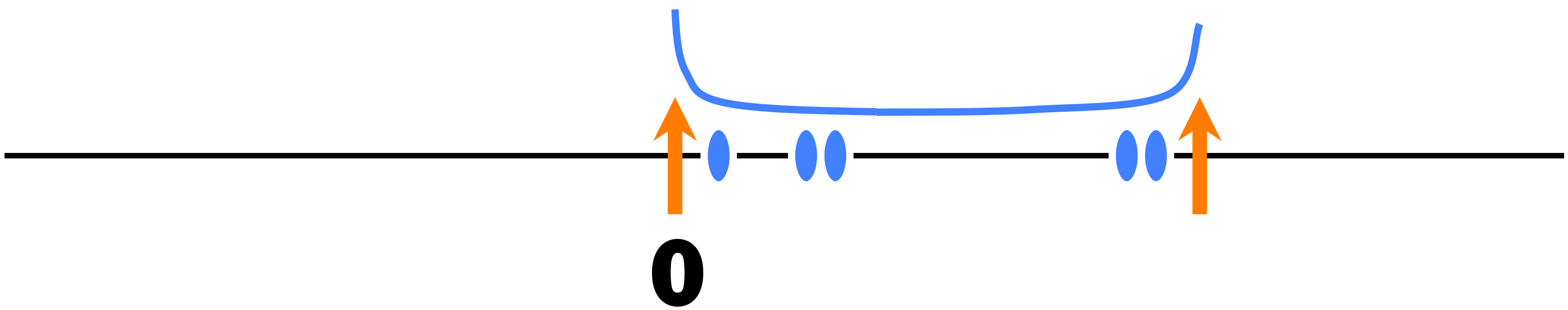
$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}$$



$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step i+1

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots$$



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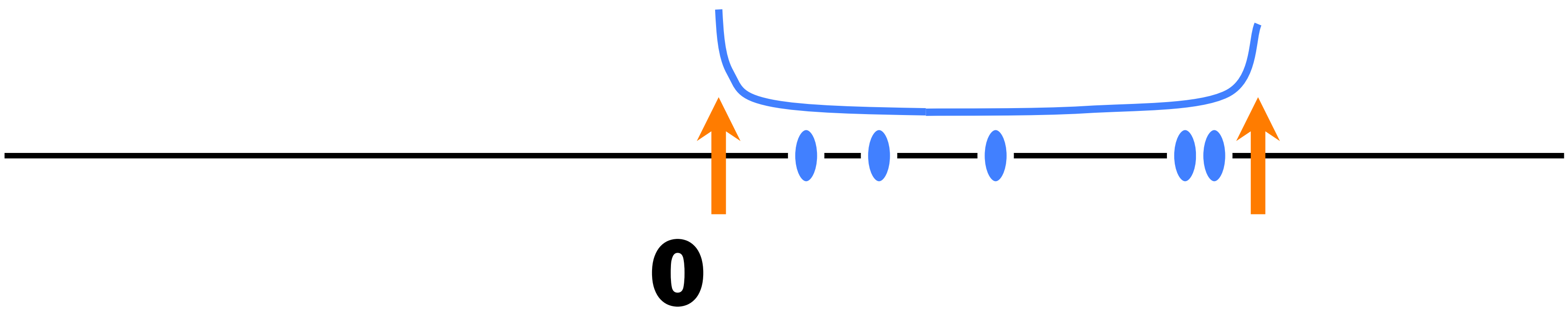

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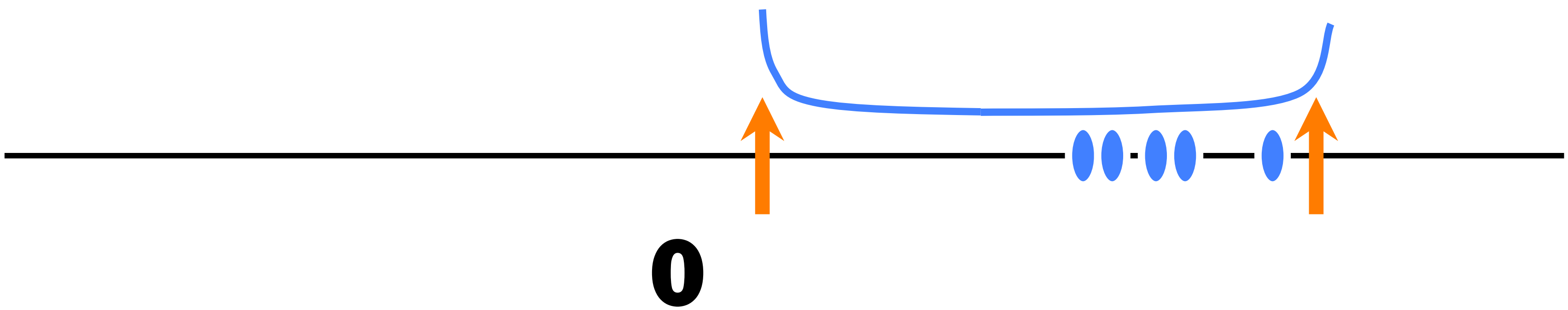


$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

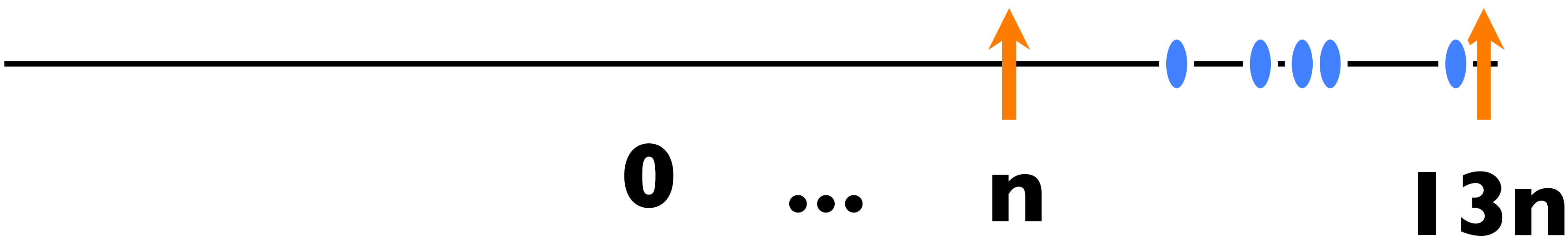
Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$

$$\Phi^u(A) \leq 1$$
$$\Phi_\ell(A) \leq 1.$$

Step 6n

$$A^{(i)}, A^{(i+1)}, A^{(i+2)}, A^{(i+3)}, \ldots, A^{(6n)}$$



2.6-approximation with 6n vectors.

□

Twice-Ramanujan

Fixing dn steps and tightening parameters gives ratio

$$\frac{\lambda_{max}(A)}{\lambda_{min}(A)} \leq \frac{d+1+2\sqrt{d}}{d+1-2\sqrt{d}}$$

Less than twice as many edges as used by Ramanujan Expander of same quality

Newman-Rabinovich:

Cut dimension of ℓ_1-metrics

Schechtman:

Embedding k-dim subspaces of L_p into ℓ_p^n

Nitzan-Olevskii-Ulanovskii:

Sampling sequences for Paley-Wiener space

S-Srivastava:

Bourgain-Tzafriri Restricted Invertibility

Srivastava-Vershynin:

Better concentration for random matrices

Open Questions

- Unweighted sparsifiers of complete graph.
- The Ramanujan bound
- Properties of vectors from graphs?
- Faster algorithm

  union of random Hamiltonian cycles?
- The Kadison-Singer Conjecture

Thank you!