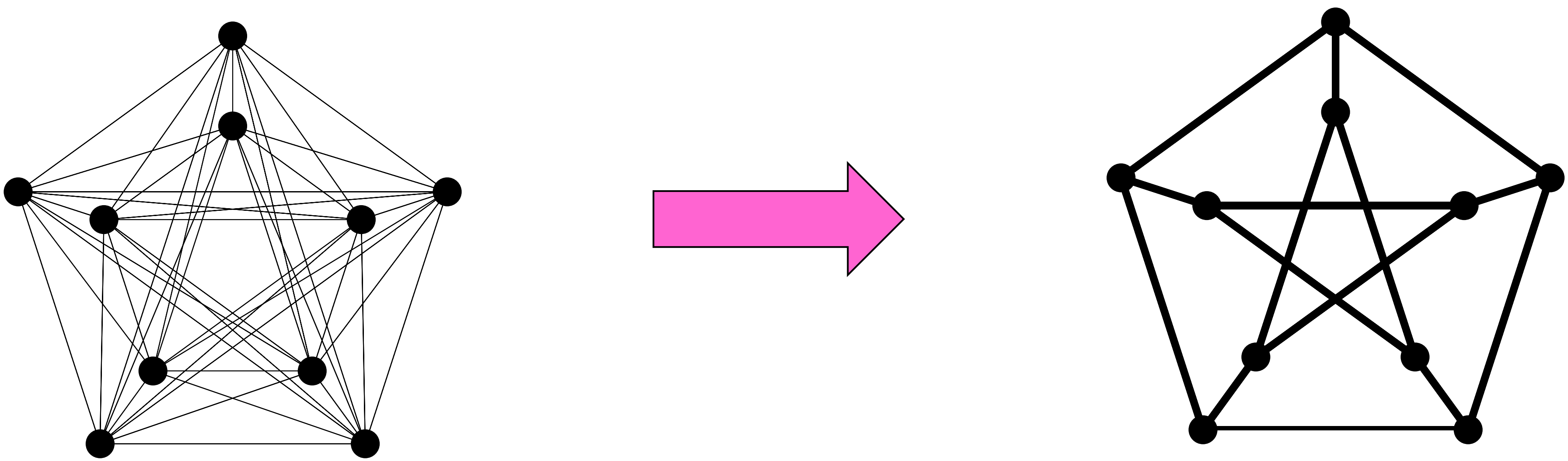
Graphs, Vectors, and Matrices

Daniel A. Spielman
Yale University



AMS Josiah Willard Gibbs Lecture

January 6, 2016

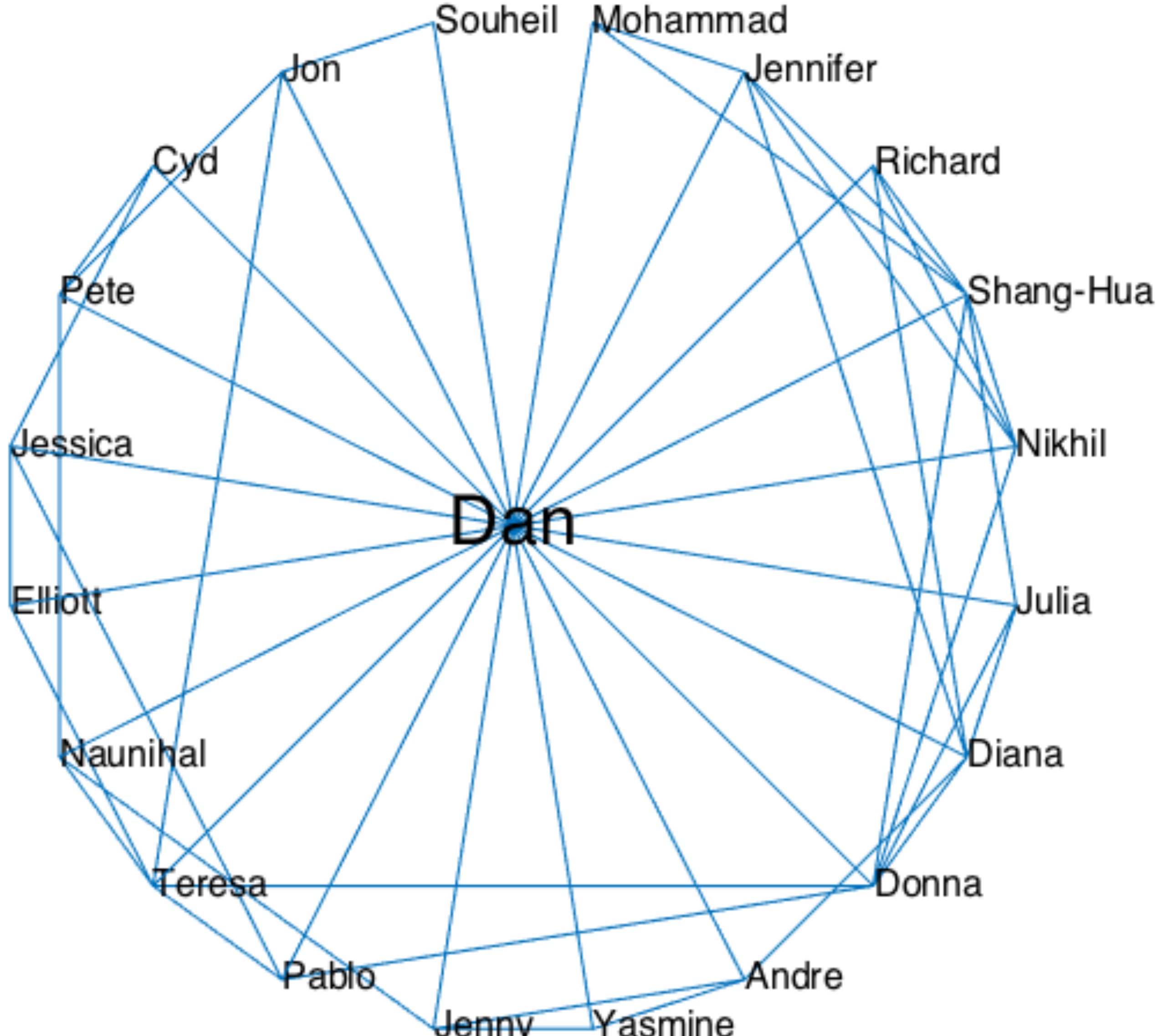
From Applied to Pure Mathematics

Algebraic and Spectral Graph Theory

Sparsification:
approximating graphs by graphs with fewer edges

The Kadison-Singer problem

A Social Network Graph



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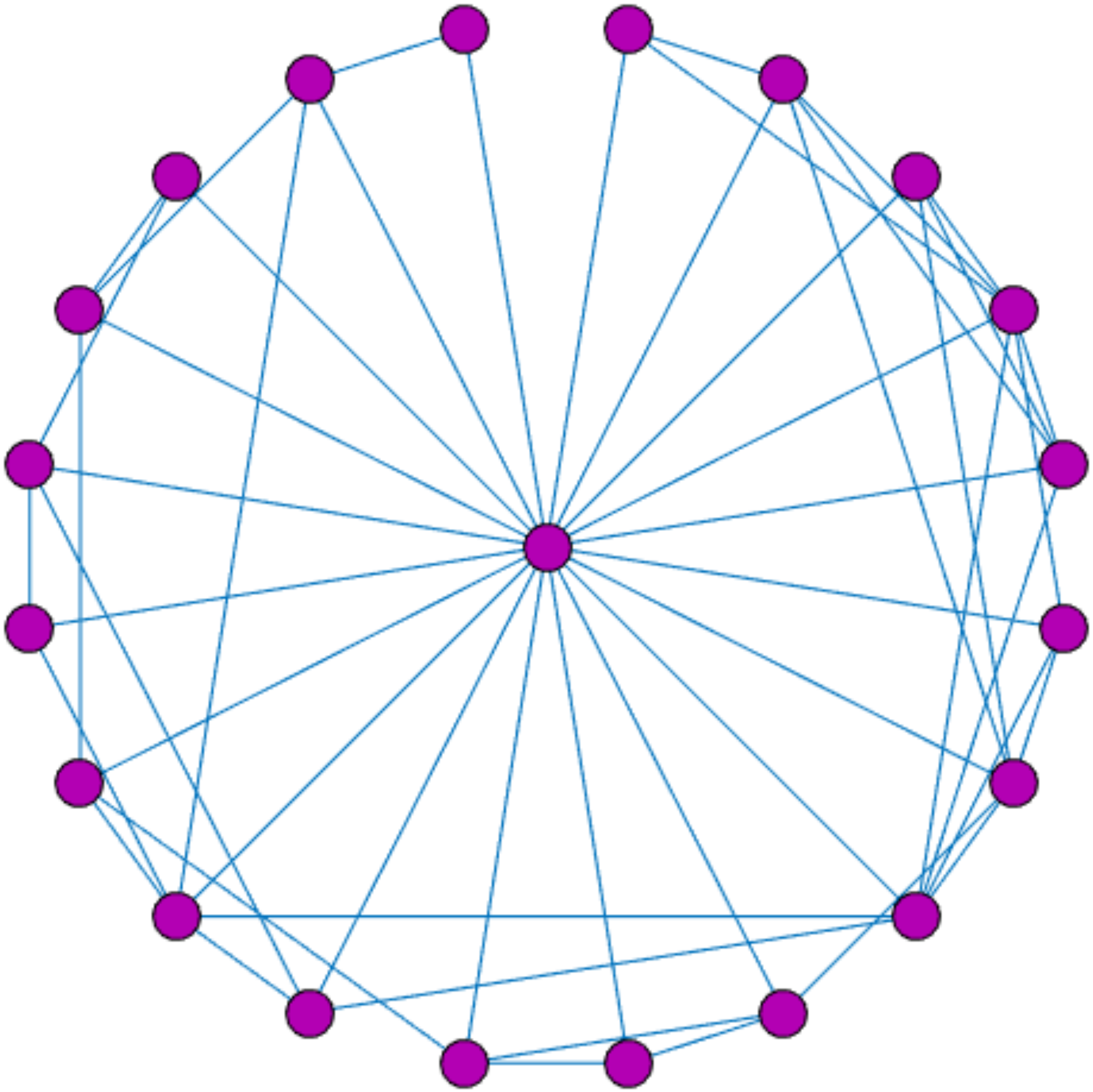

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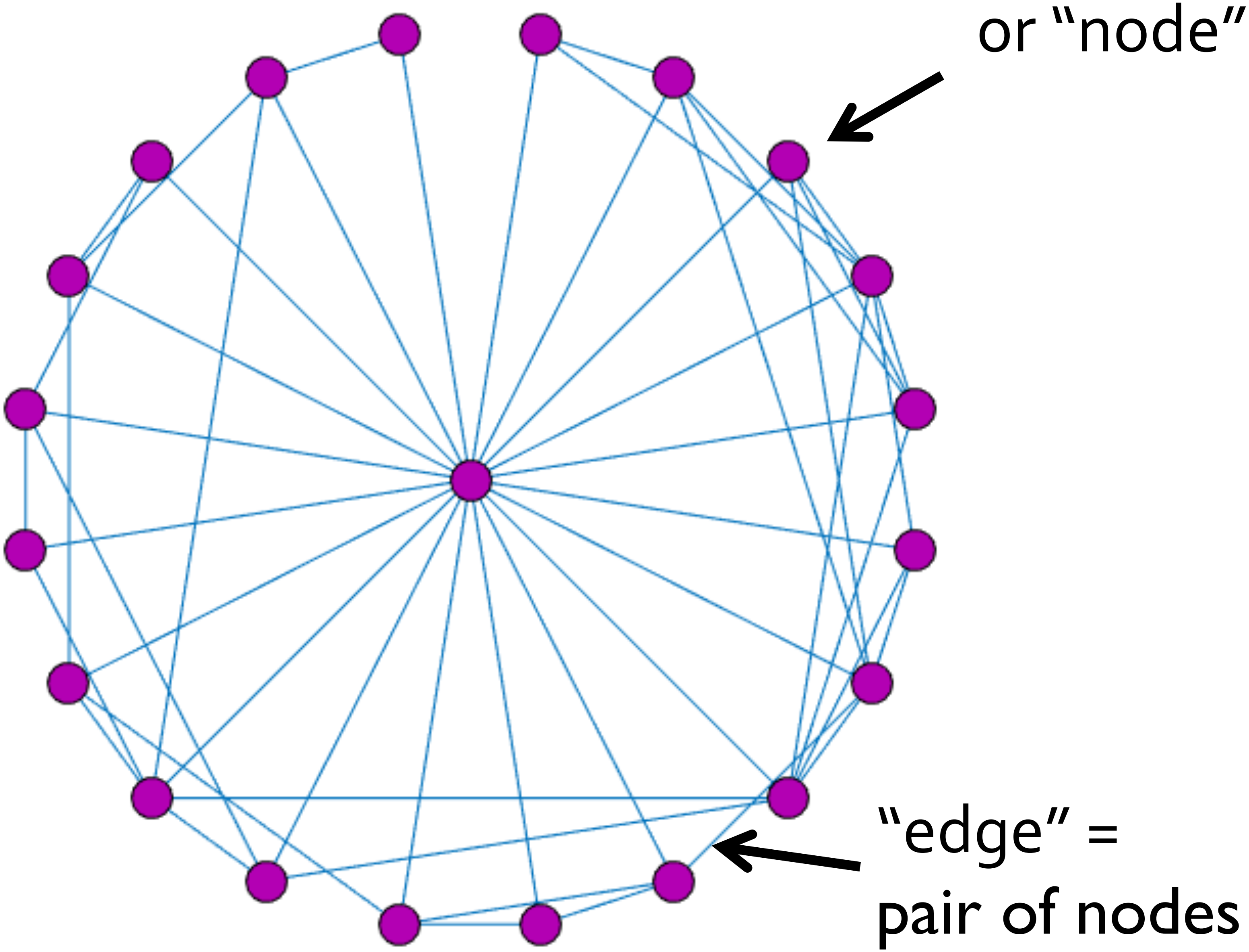

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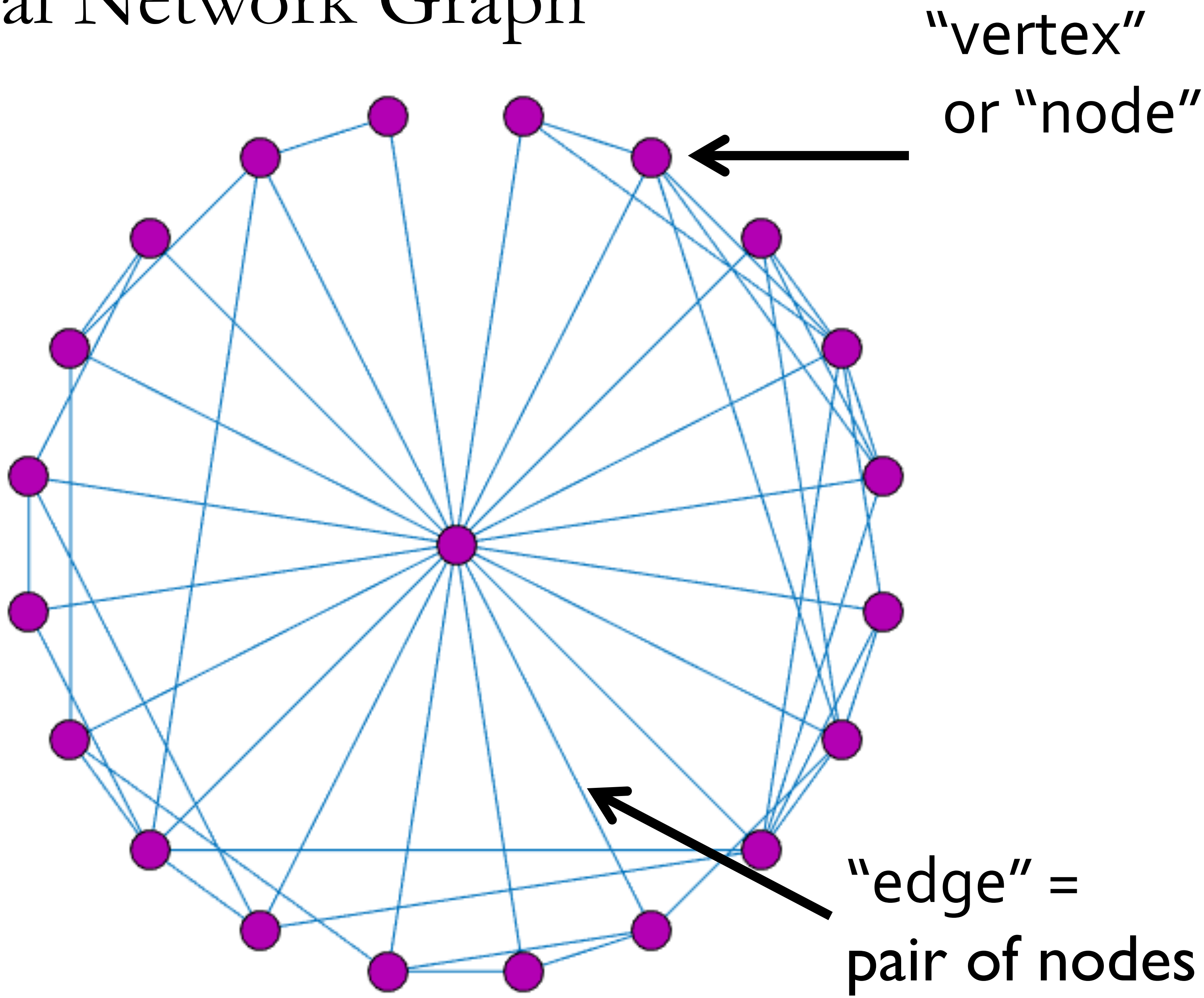
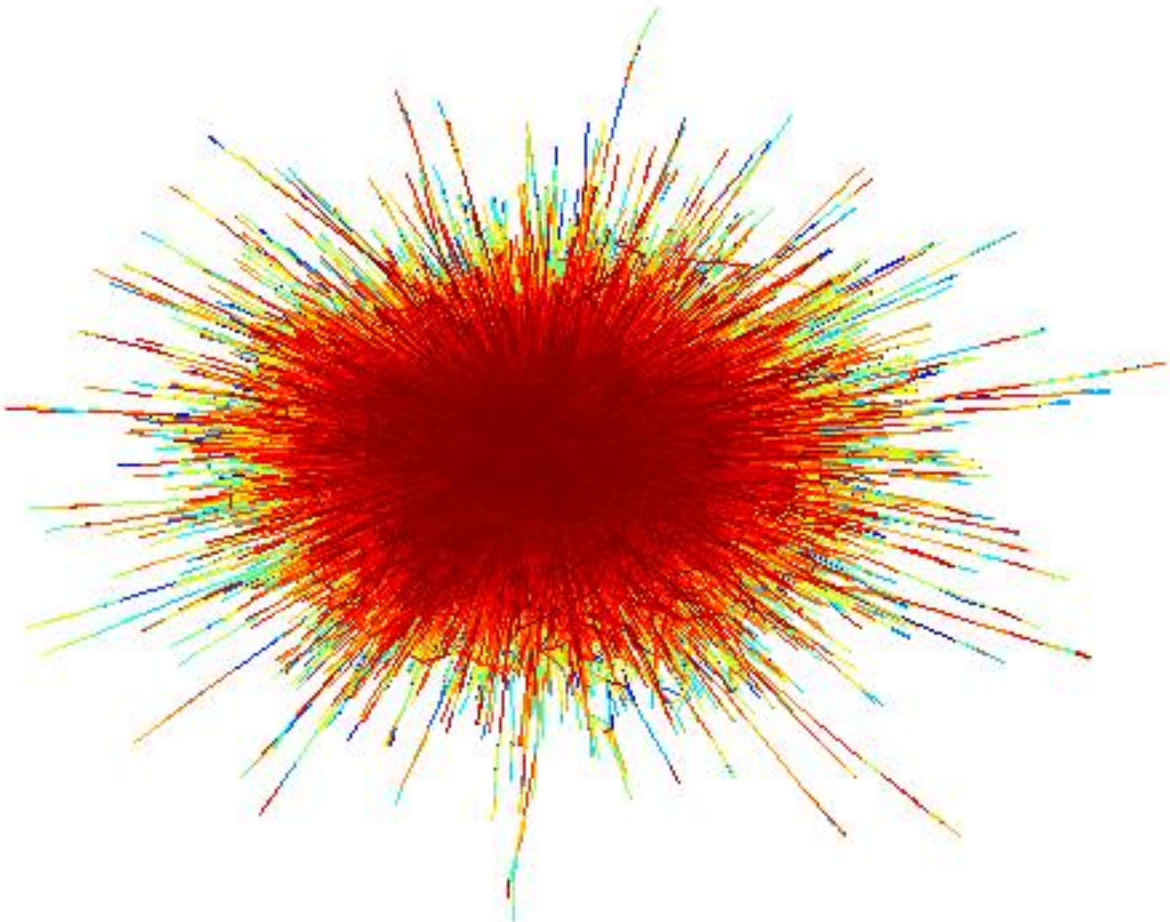
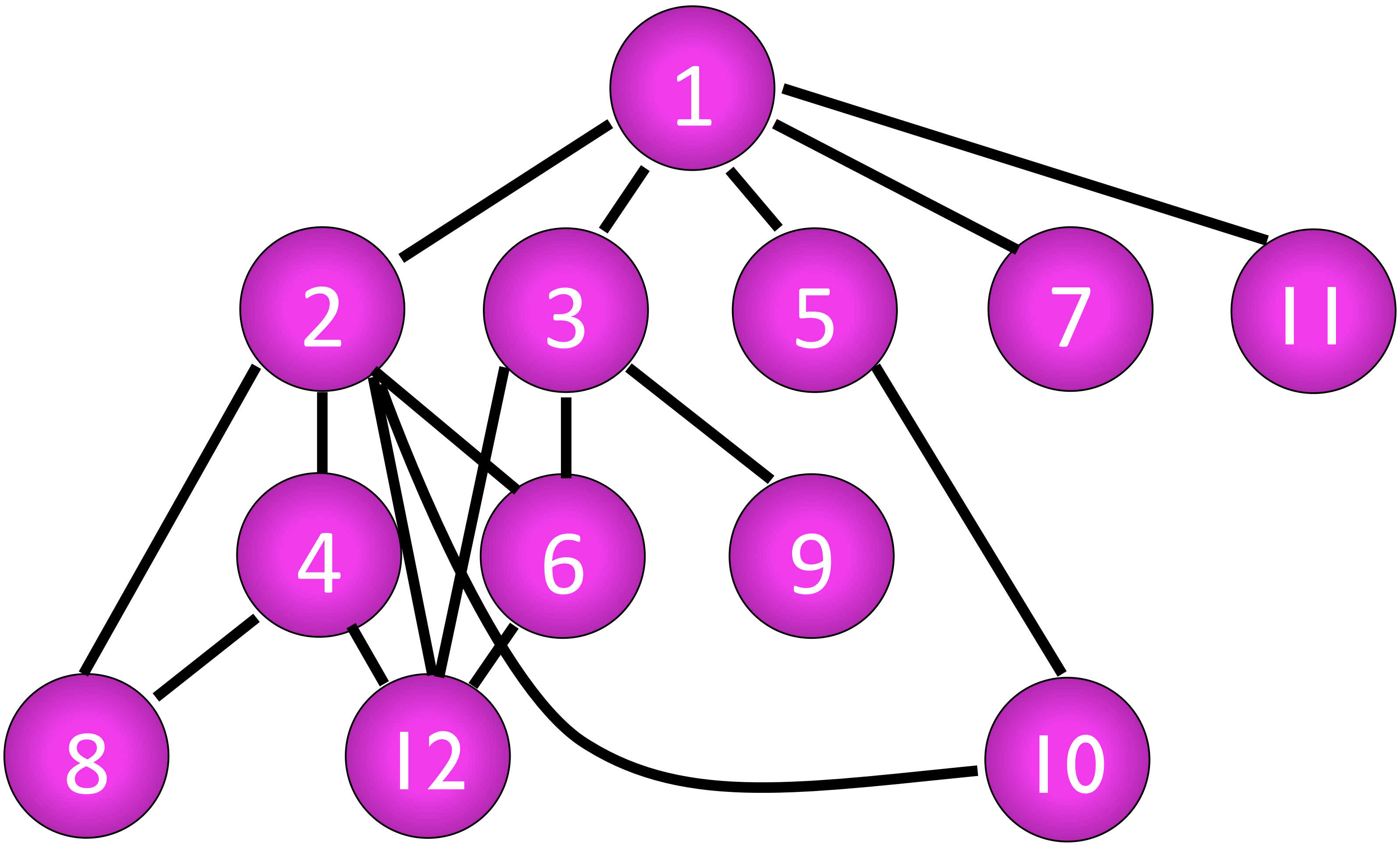


A Big Social Network Graph

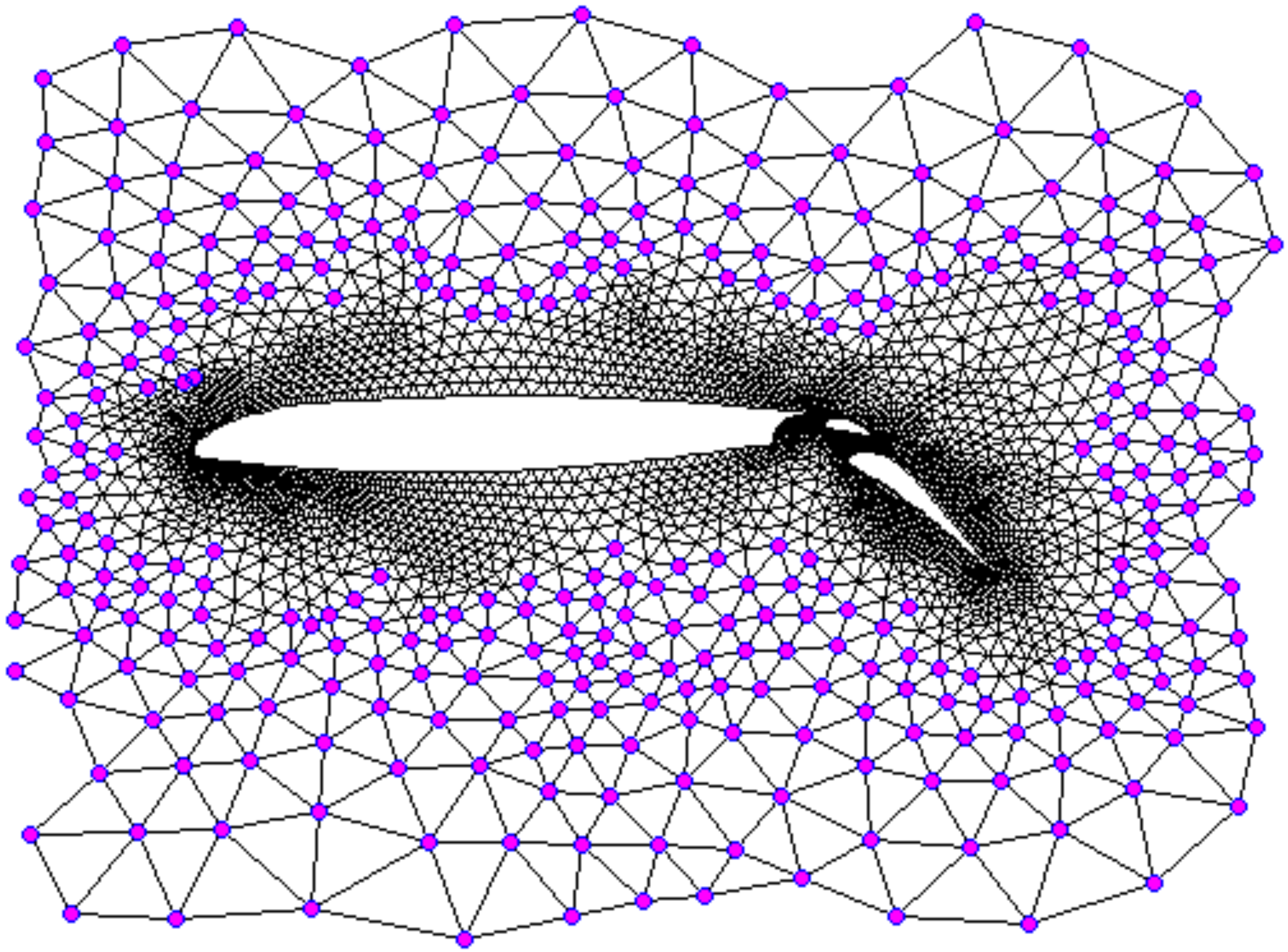


A Graph $G = (V, E)$

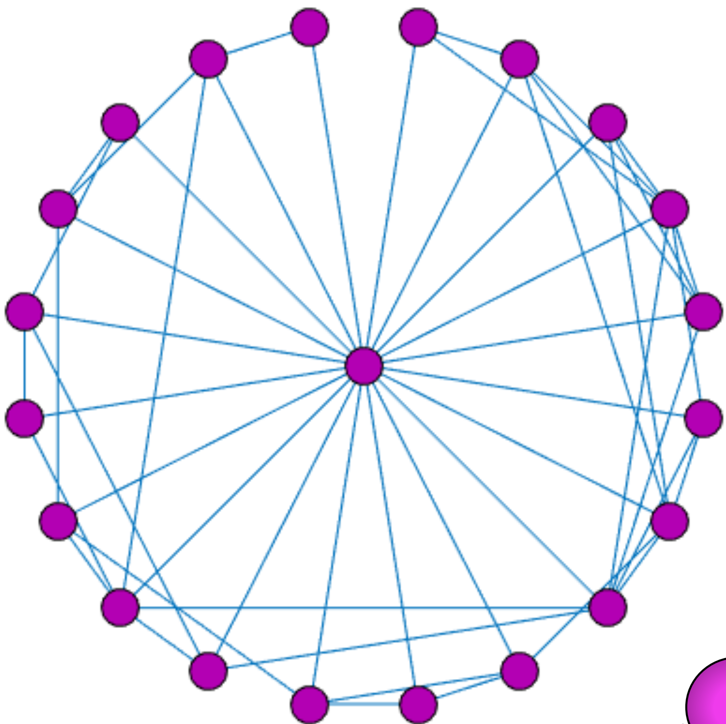
V = vertices, E = edges, pairs of vertices

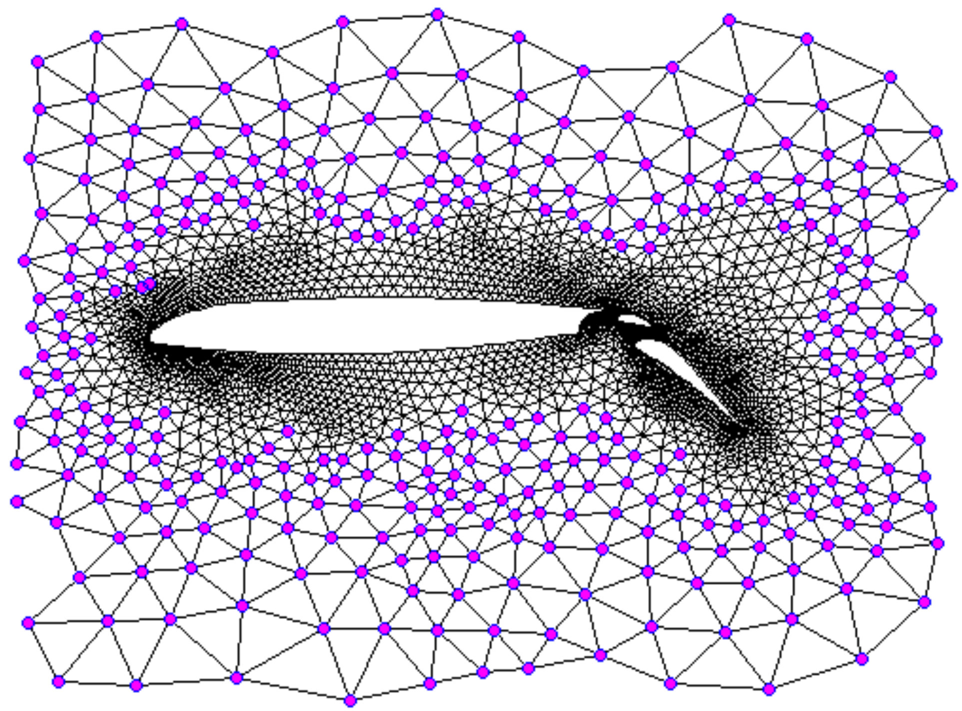


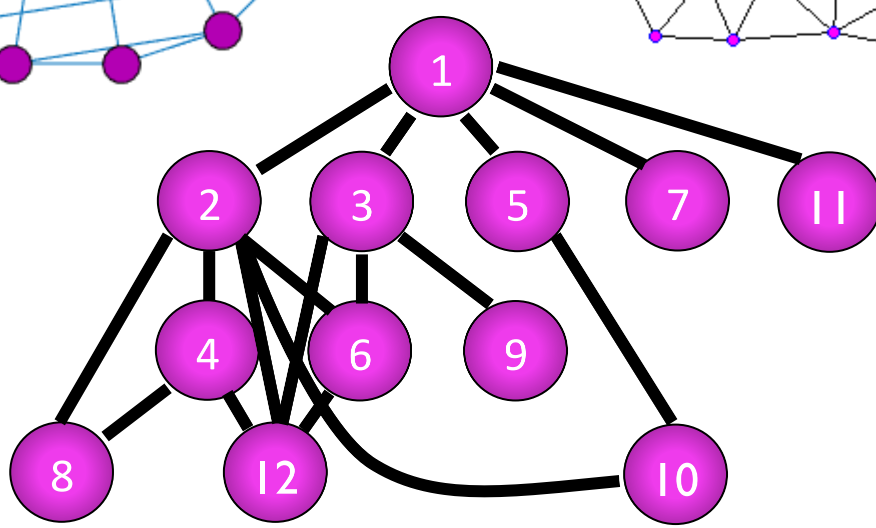
The Graph of a Mesh



Examples of Graphs







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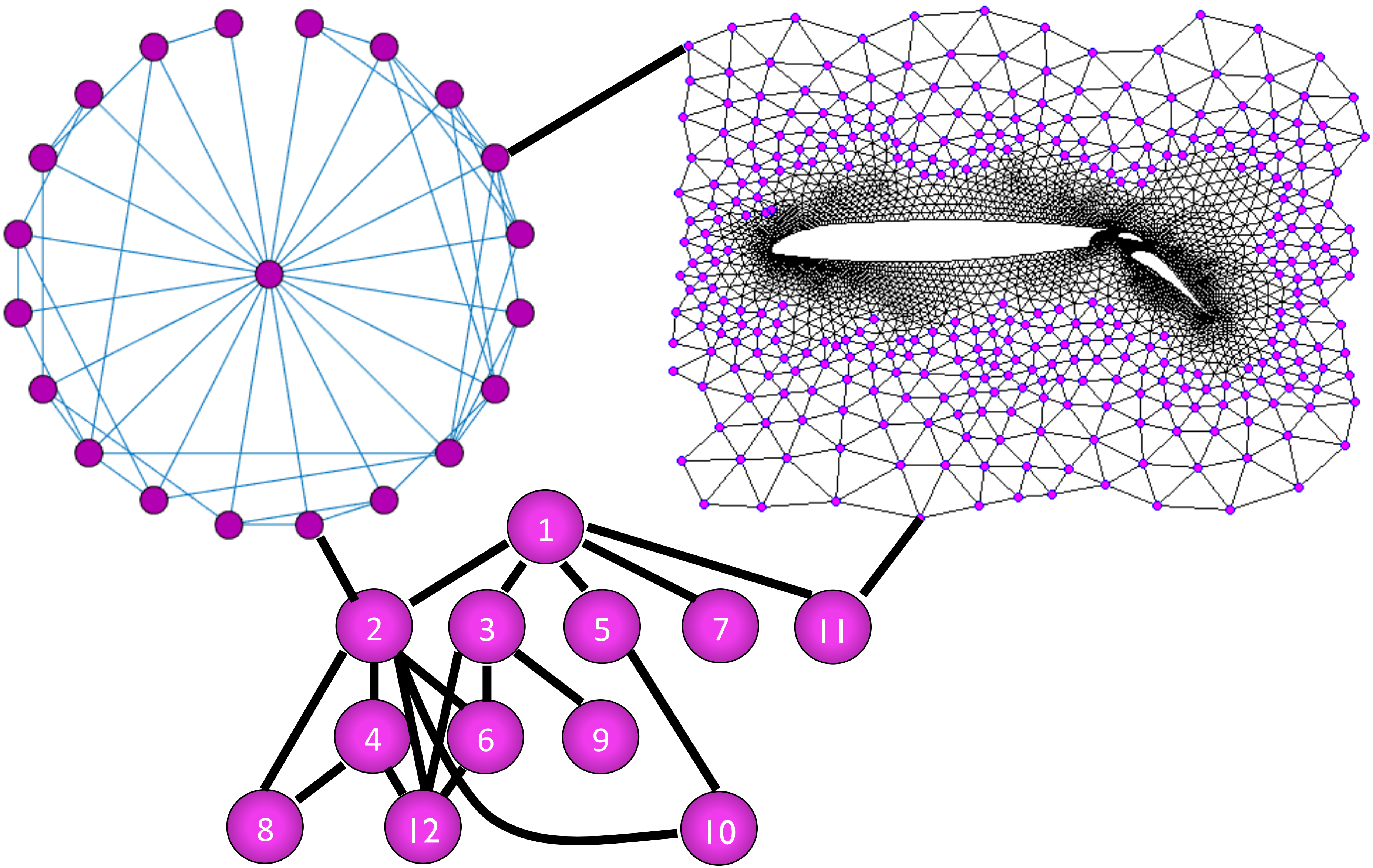


How to understand large-scale structure

Draw the graph

Identify communities and hierarchical structure

Use physical metaphors

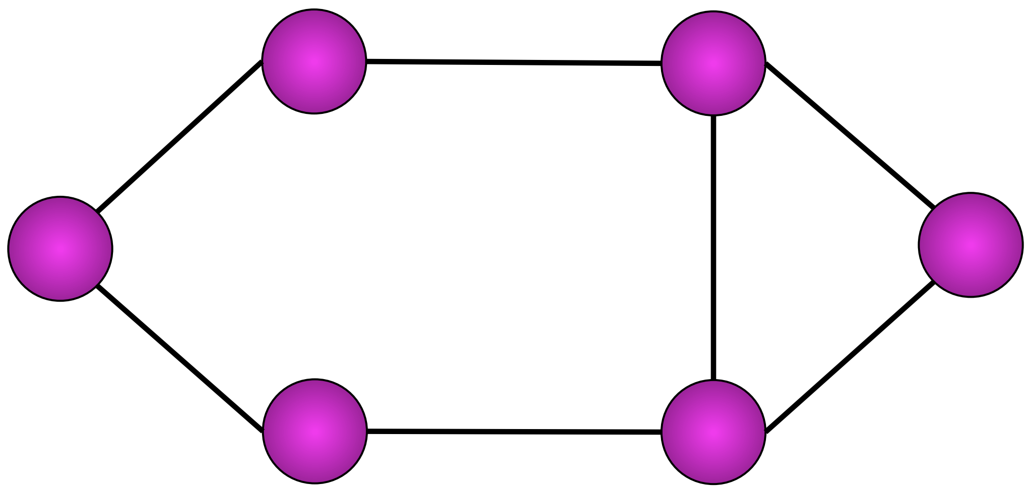
- Edges as resistors or rubber bands

Examine processes

- Diffusion of gas / Random Walks

The Laplacian quadratic form of $G = (V, E)$

$$x : V \to \mathbb{R} \qquad \sum_{(a,b) \in E} (x(a) - x(b))^2$$



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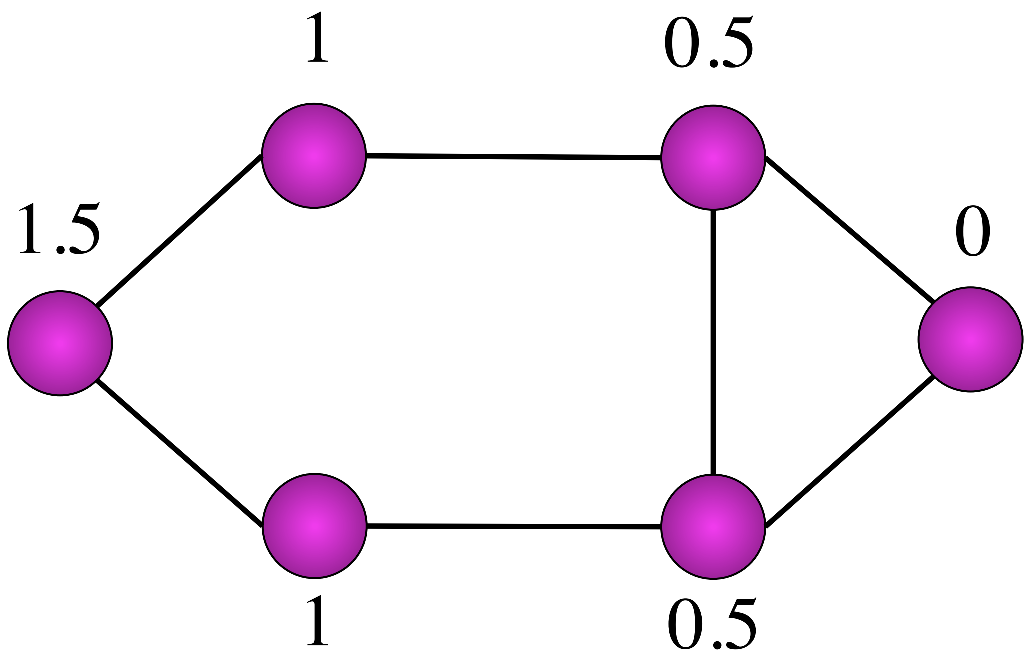

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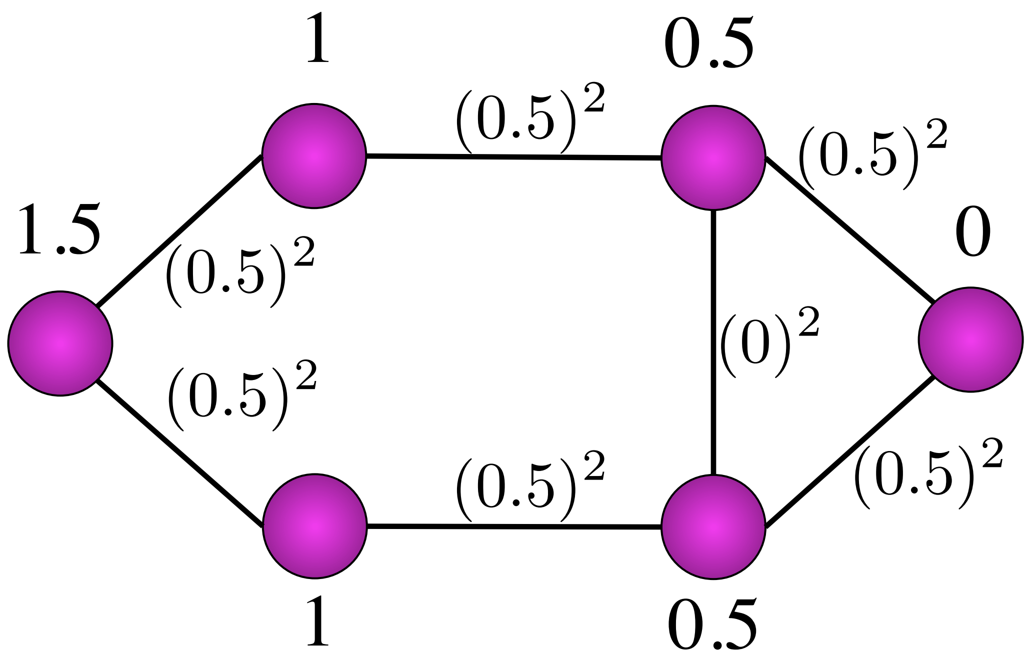


The Laplacian matrix of $G = (V, E)$

$$x : V \to \mathbb{R}$$

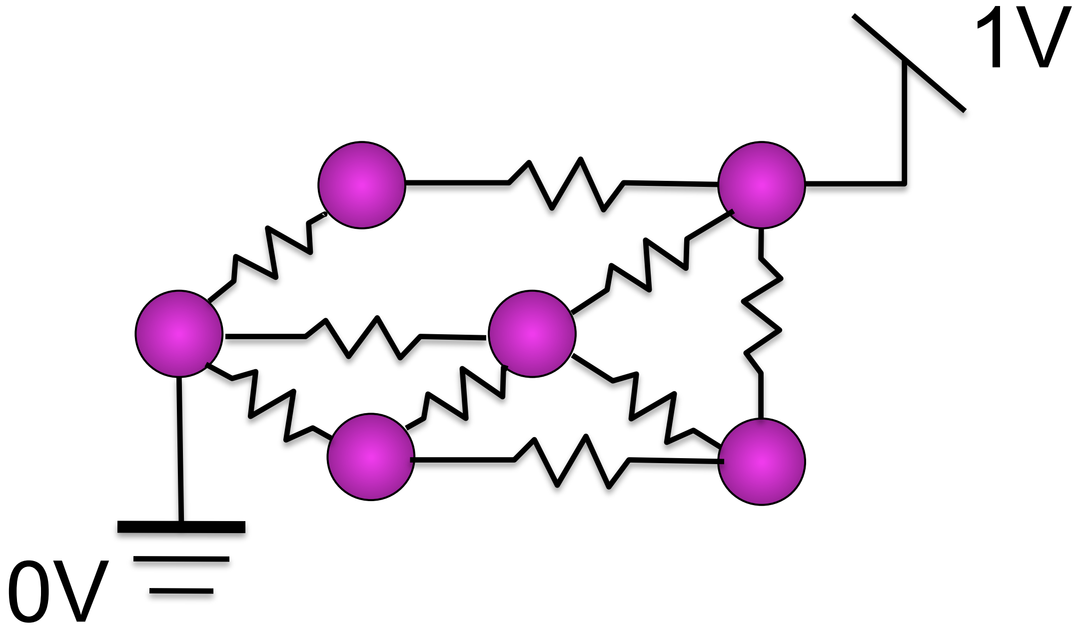
$$\sum_{(a,b) \in E} (x(a) - x(b))^2$$

$$= x^T L x$$

Graphs as Resistor Networks

View edges as resistors connecting vertices

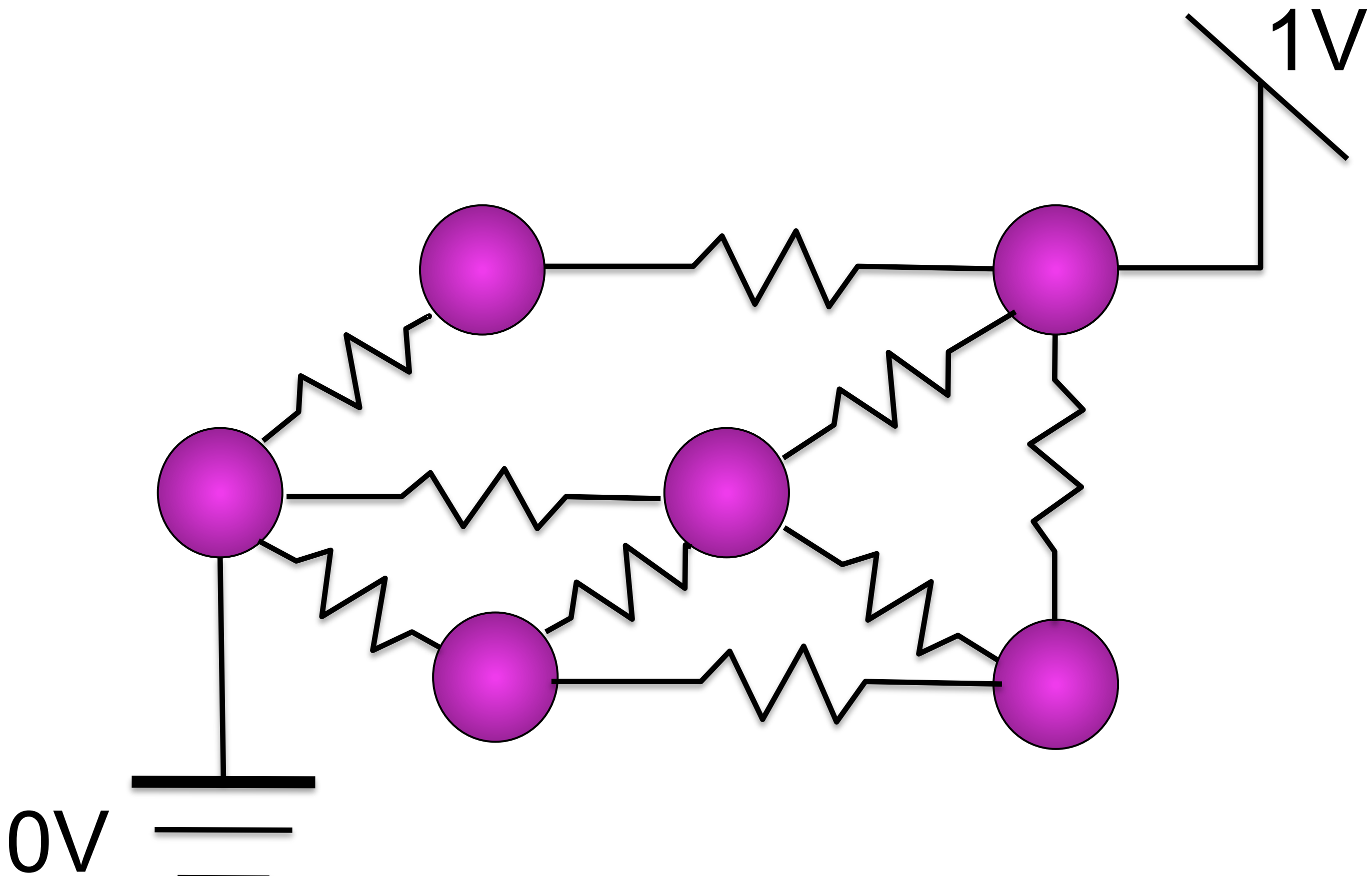
Apply voltages at some vertices.
Measure induced voltages and current flow.



Graphs as Resistor Networks

Induced voltages minimize subject to constraints.

$$\sum_{(a,b)\in E} (x(a) - x(b))^2,$$



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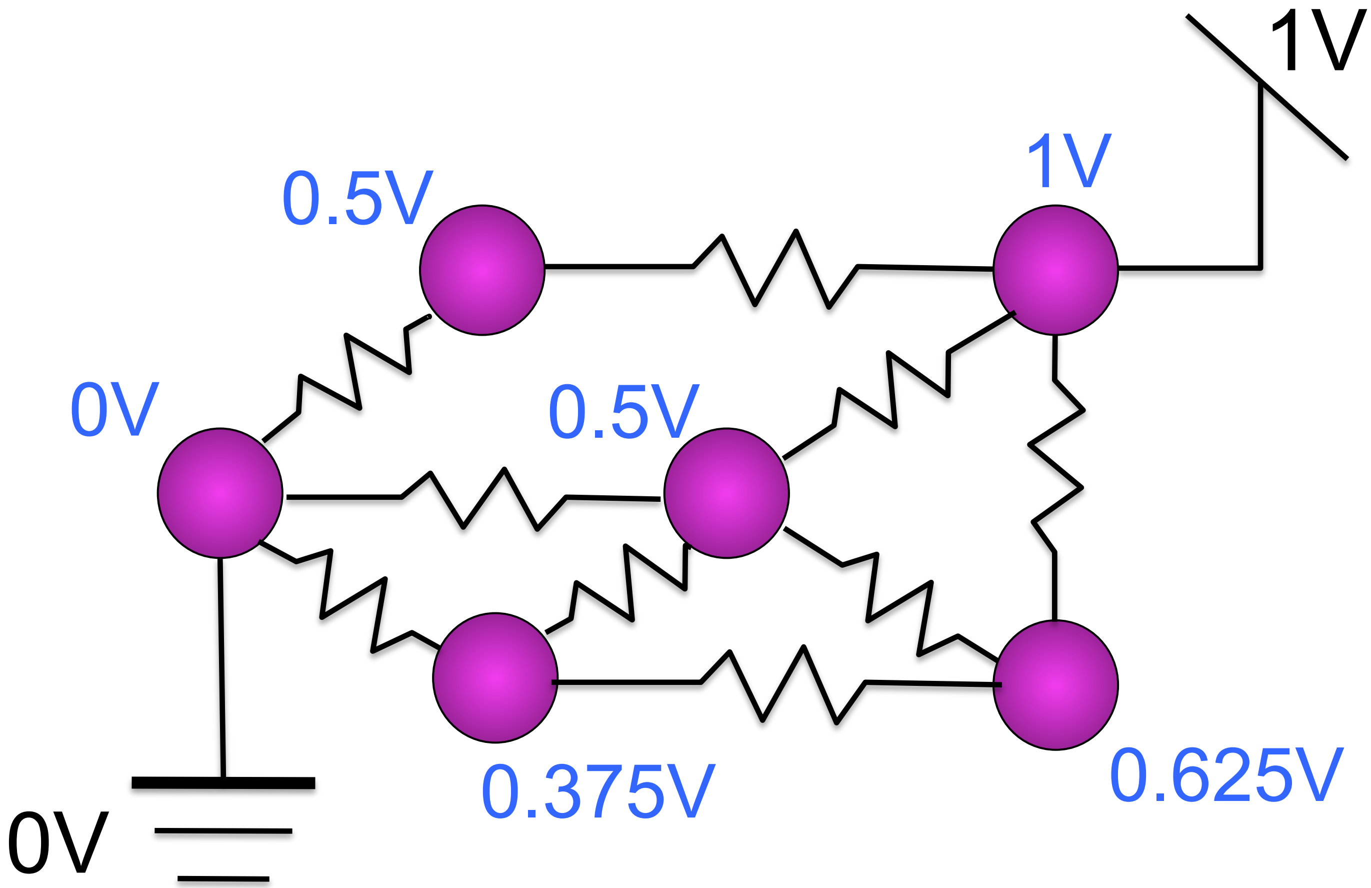
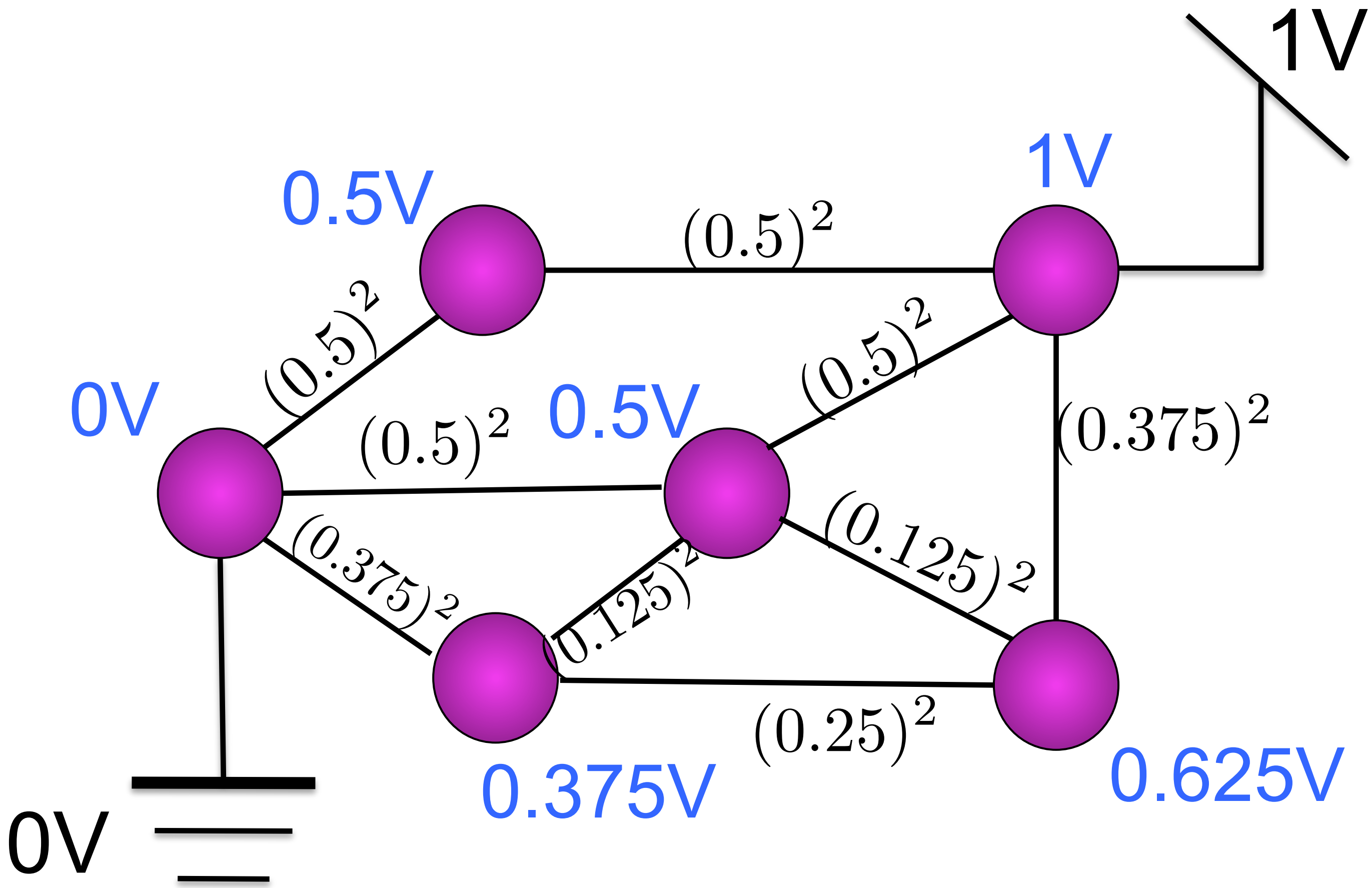


Graphs as Resistor Networks

Induced voltages minimize $\sum_{(a,b)\in E} (x(a) - x(b))^2$, subject to constraints.

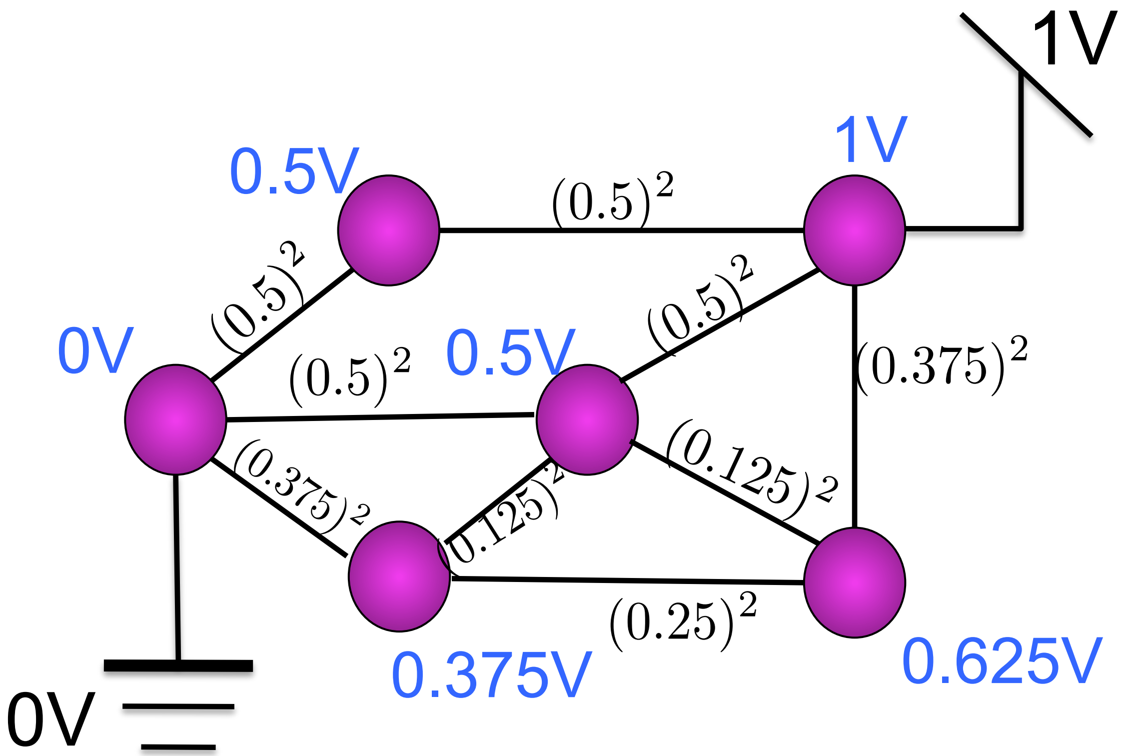


Graphs as Resistor Networks

Induced voltages minimize subject to constraints.

$$\sum_{(a,b)\in E} (x(a) - x(b))^2,$$

Effective conductance = current flow with one volt



Weighted Graphs

Edge (a, b) assigned a non-negative real weight $w_{a,b} \in \mathbb{R}$ measuring

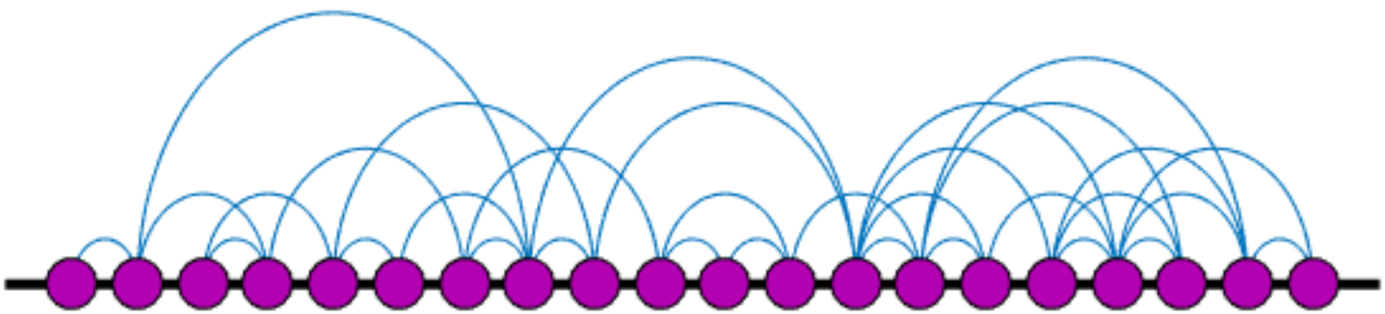
strength of connection

1/resistance

$$x^T L x = \sum_{(a,b) \in E} w_{a,b} (x(a) - x(b))^2$$

Spectral Graph Drawing (Hall '70)

Want to map $V \to \mathbb{R}$ with most edges short



Edges are drawn as curves for visibility.

Spectral Graph Drawing (Hall '70)

Want to map $V \to \mathbb{R}$ with most edges short

Minimize $$x^T L x = \sum_{(a,b) \in E} (x(a) - x(b))^2$$

to fix scale, require $$\sum_a x(a)^2 = 1$$

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to fix scale, require $$\sum_a x(a)^2 = 1$$

$$\|x\| = 1$$

Courant-Fischer Theorem

$$\lambda_1 = \min_{\substack{x \neq 0 \\ \|x\|=1}} x^T L x \qquad\qquad v_1 = \arg \min_{\substack{x \neq 0 \\ \|x\|=1}} x^T L x$$

Where λ_1 is the smallest eigenvalue of L and v_1 is the corresponding eigenvector.

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Where λ_1 is the smallest eigenvalue of L and v_1 is the corresponding eigenvector.

For $x^T L x = \sum_{(a,b) \in E} (x(a) - x(b))^2$

$\lambda_1 = 0$ and v_1 is a constant vector

Spectral Graph Drawing (Hall '70)

Want to map $V \to \mathbb{R}$ with most edges short

Minimize $x^T L x = \sum_{(a,b) \in E} (x(a) - x(b))^2$

Such that $\|x\| = 1$ and $\sum_a x(a) = 0$

Spectral Graph Drawing (Hall '70)

Want to map $V \to \mathbb{R}$ with most edges short

Minimize $x^T L x = \sum_{(a,b) \in E} (x(a) - x(b))^2$

Such that $\|x\| = 1$ and $\sum_a x(a) = 0$

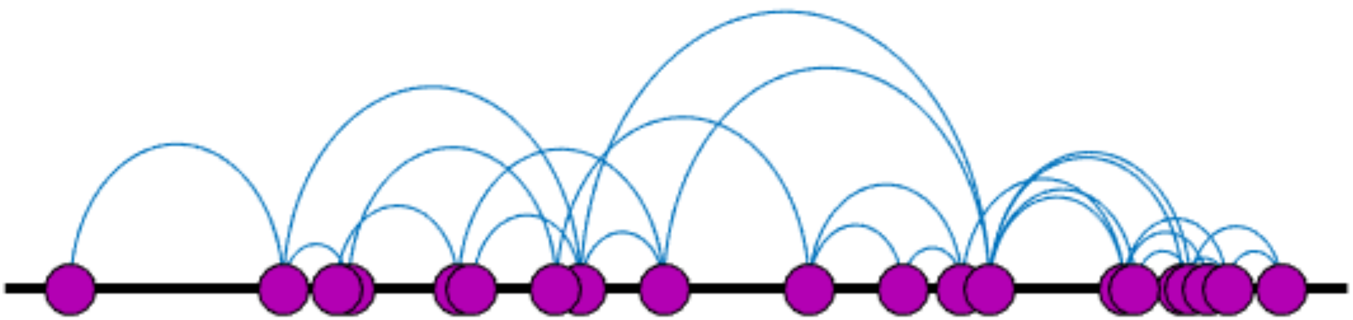
Courant-Fischer Theorem:

solution is v_2, the eigenvector of λ_2 ,

the second-smallest eigenvalue

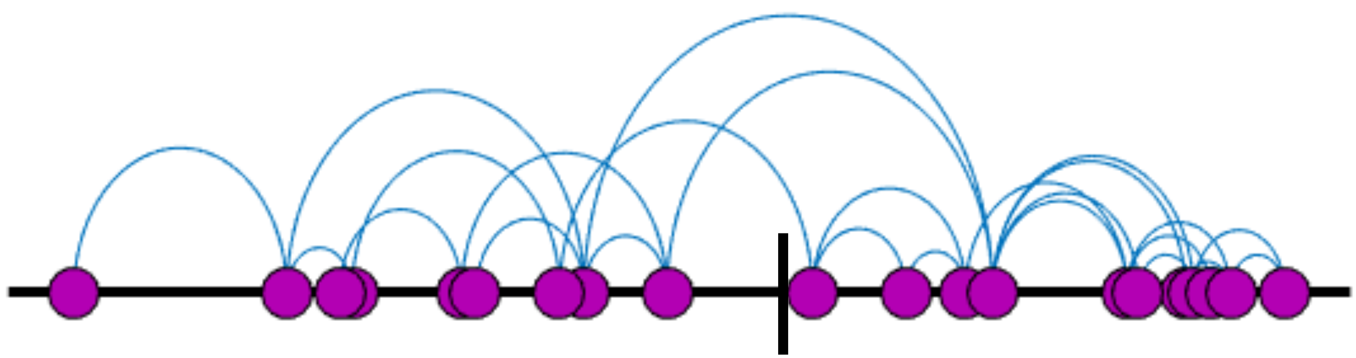
Spectral Graph Drawing (Hall '70)

$$\sum_{(a,b)\in E} (x(a) - x(b))^2 \quad = \text{area under blue curves}$$



Spectral Graph Drawing (Hall '70)

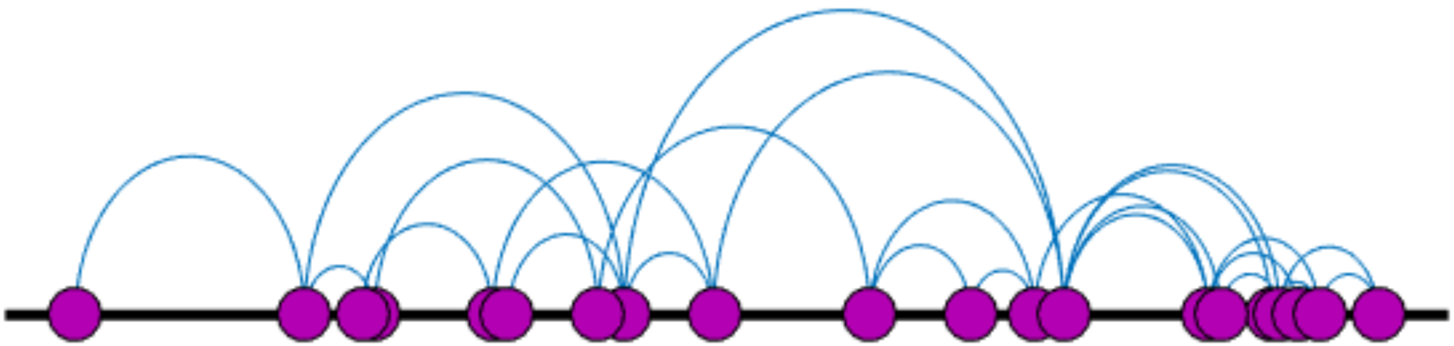
$$\sum_{(a,b)\in E} (x(a)-x(b))^2 \quad = \text{area under blue curves}$$



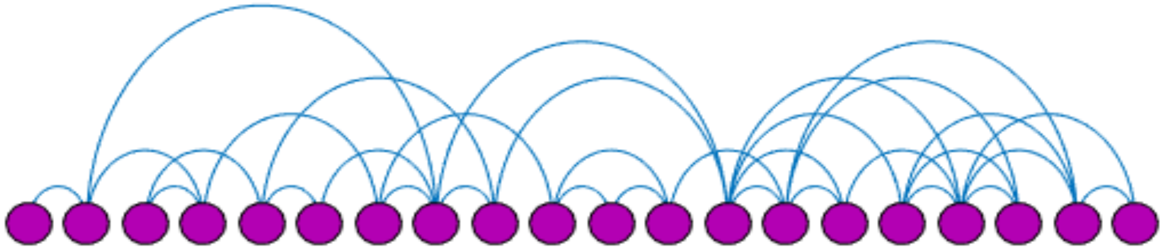
$$\|x\| = 1$$

$$0 = \sum_a x(a)$$

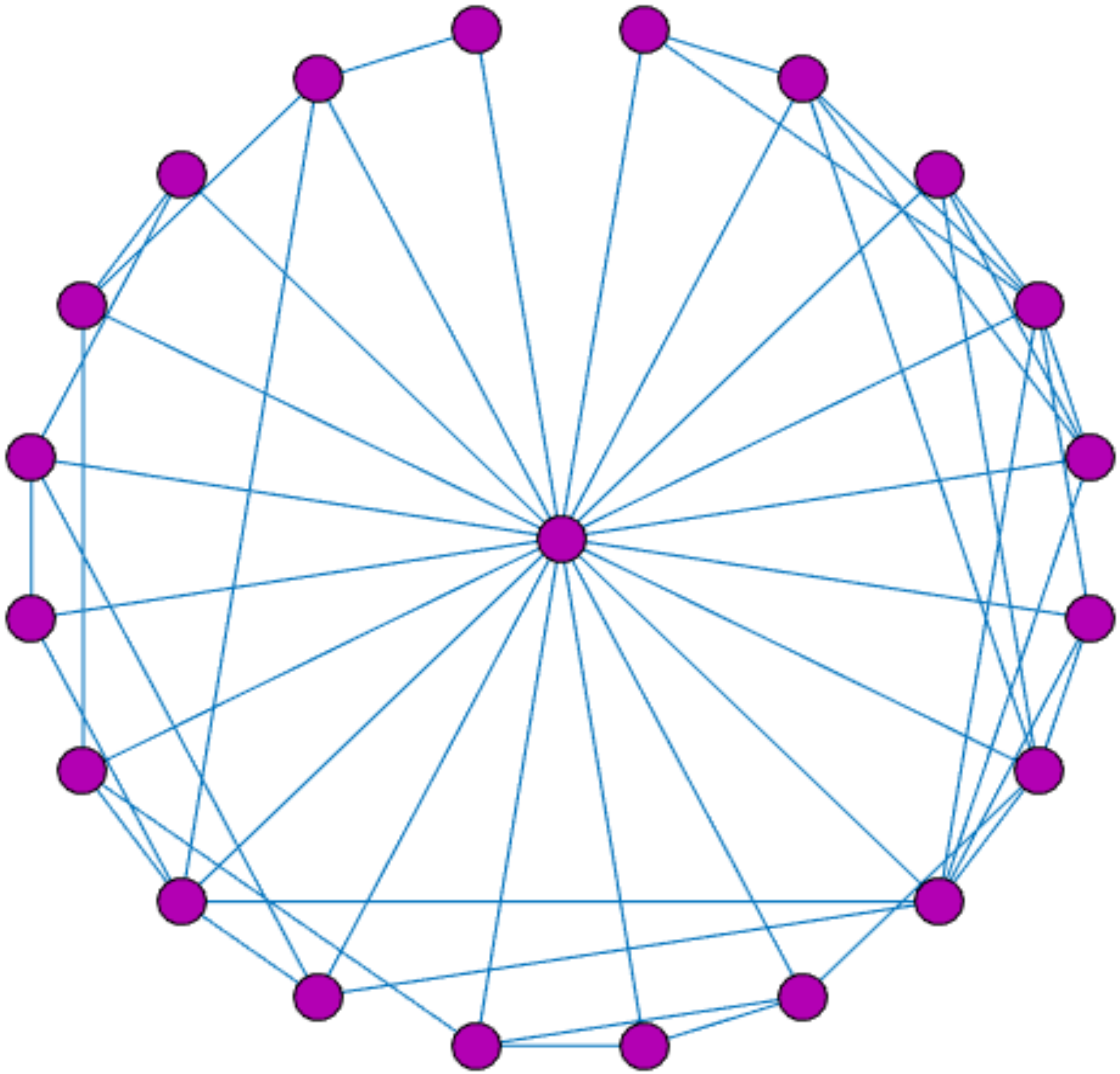
Space the points evenly



And, move them to the circle



Finish by putting me back in the center



Spectral Graph Drawing (Hall '70)

Want to map $V \to \mathbb{R}^2$ with most edges short

Minimize $\displaystyle\sum_{(a,b)\in E} \left\| \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} - \begin{pmatrix} x(b) \\ y(b) \end{pmatrix} \right\|^2$

Such that $\|x\| = 1$ and $\displaystyle\sum_a x(a) = 0$

$\|y\| = 1$ and $\displaystyle\sum_a y(a) = 0$

Spectral Graph Drawing (Hall '70)

Want to map $V \to \mathbb{R}^2$ with most edges short

Minimize $\displaystyle\sum_{(a,b)\in E} \left\| \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} - \begin{pmatrix} x(b) \\ y(b) \end{pmatrix} \right\|^2$

Such that $\|x\| = 1$ and $1^T x = 0$

$\|y\| = 1$ and $1^T y = 0$

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Want to map $V \to \mathbb{R}^2$ with most edges short

Minimize $\displaystyle\sum_{(a,b)\in E} \left\| \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} - \begin{pmatrix} x(b) \\ y(b) \end{pmatrix} \right\|^2$

Such that $\|x\| = 1$ and $1^T x = 0$

$\|y\| = 1$ and $1^T y = 0$

and $x^T y = 0$, to prevent $x = y$

Spectral Graph Drawing (Hall '70)

Minimize $$\sum_{(a,b)\in E} \left\| \begin{pmatrix} x(a) \\ y(a) \end{pmatrix} - \begin{pmatrix} x(b) \\ y(b) \end{pmatrix} \right\|^2$$

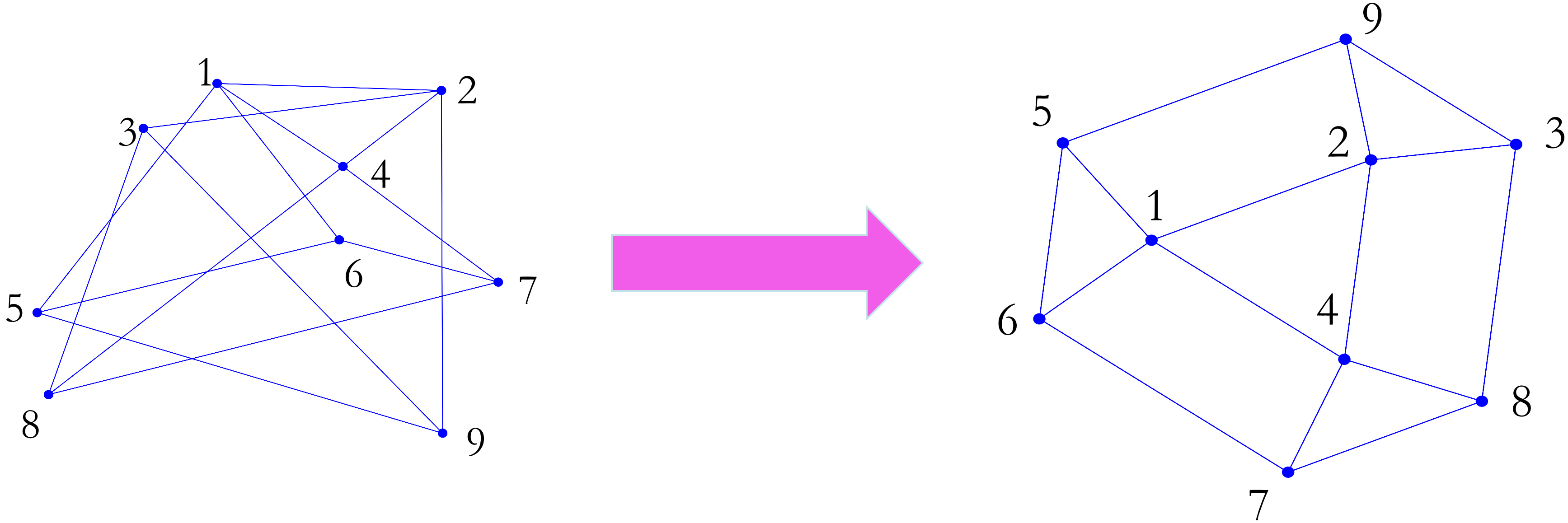
Such that $\|x\| = 1 \quad \|y\| = 1$

$$1^T x = 0 \quad 1^T y = 0 \quad \text{and} \quad x^T y = 0$$

Courant-Fischer Theorem:

solution is $x = v_2, y = v_3$, up to rotation

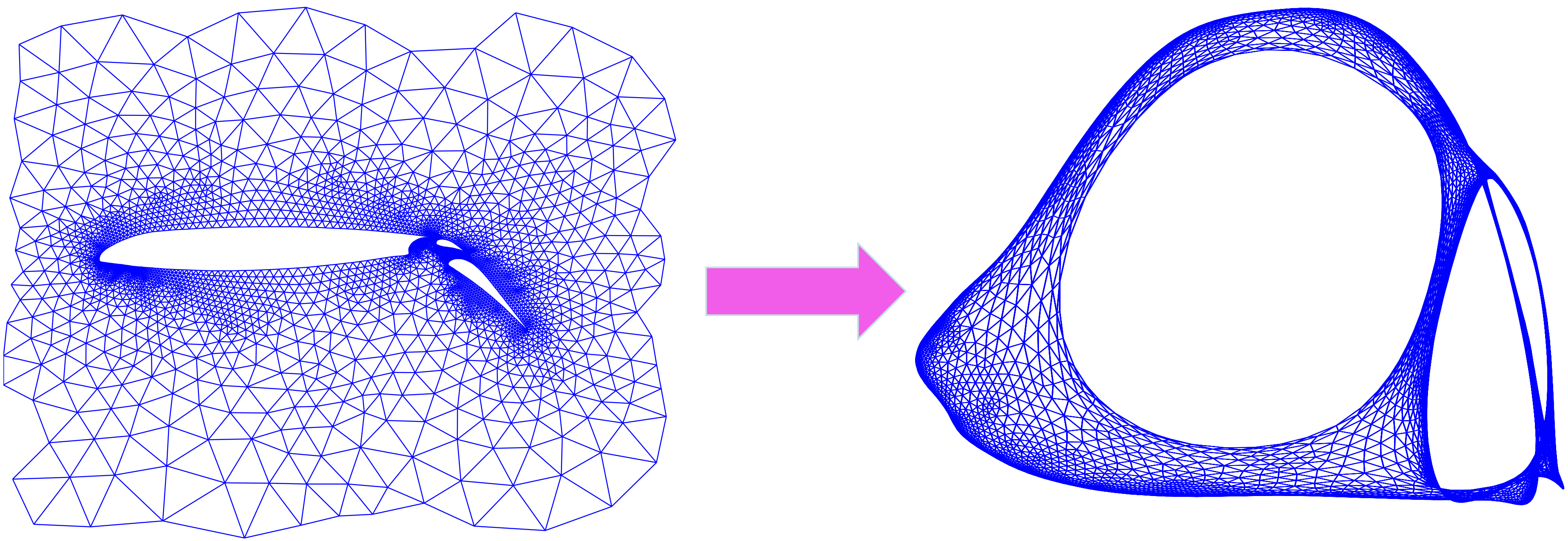
Spectral Graph Drawing (Hall '70)



Arbitrary Drawing

Spectral Drawing

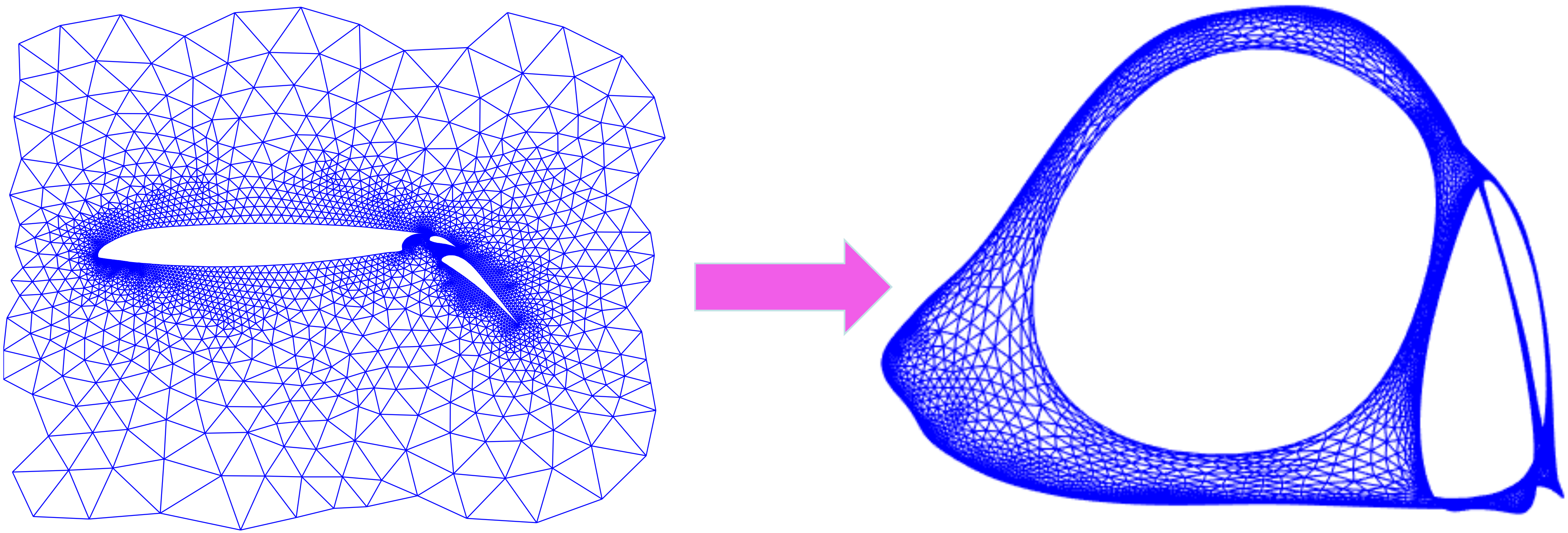
Spectral Graph Drawing (Hall '70)



Original Drawing

Spectral Drawing

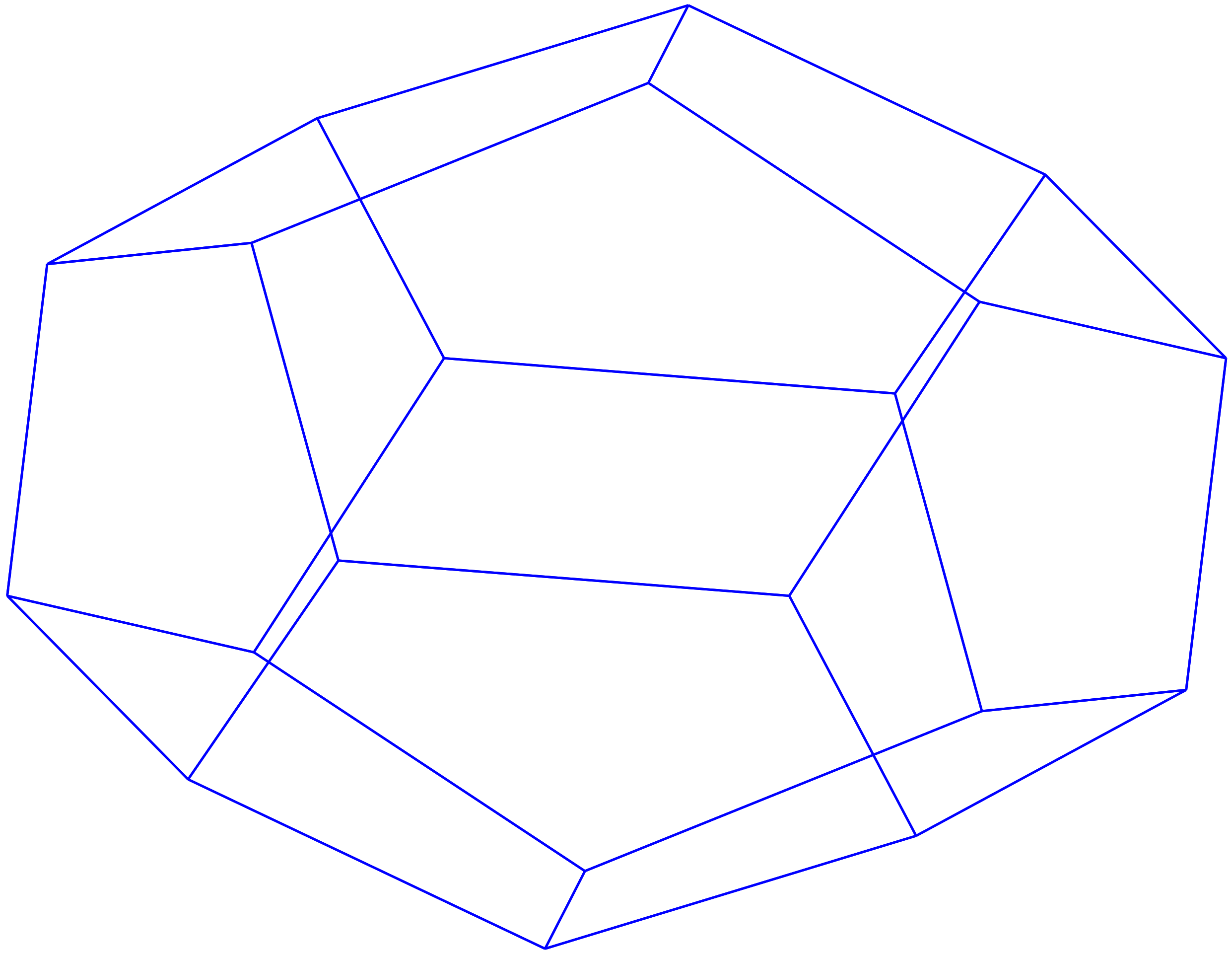
Spectral Graph Drawing (Hall '70)



Original Drawing

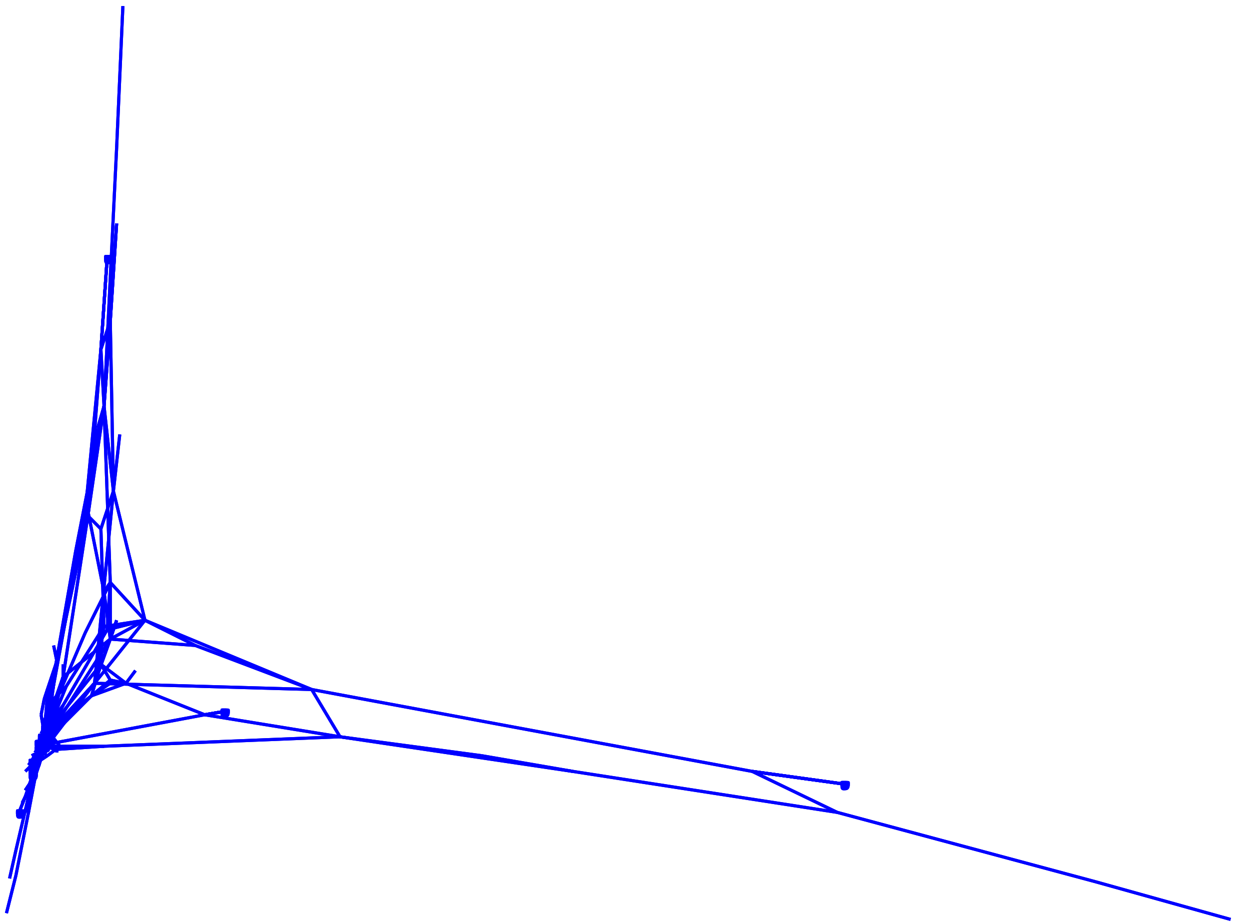
Spectral Drawing

Dodecahedron



Best embedded by first three eigenvectors

Spectral drawing of Erdos graph: edge between co-authors of papers



When there is a "nice" drawing:

Most edges are short
Vertices are spread out and don't clump too much

⟹ λ_2 is close to 0

When λ_2 is big, say $> 10/|V|^{1/2}$
there is no nice picture of the graph

Expanders: when λ_2 is big

Formally: infinite families of graphs
of constant degree d and large λ_2

Examples: random d-regular graphs
Ramanujan graphs

Have no communities or clusters.

Incredibly useful in Computer Science:
Act like random graphs (pseudo-random)
Used in many important theorems and algorithms

Good Expander Graphs

d-regular graphs with $\lambda_2, \ldots, \lambda_n \approx d$

Courant-Fischer: for all $\begin{matrix} 1^T x = 0 \\ \|x\| = 1 \end{matrix}$ $\quad x^T L_G x \approx d$

Good Expander Graphs

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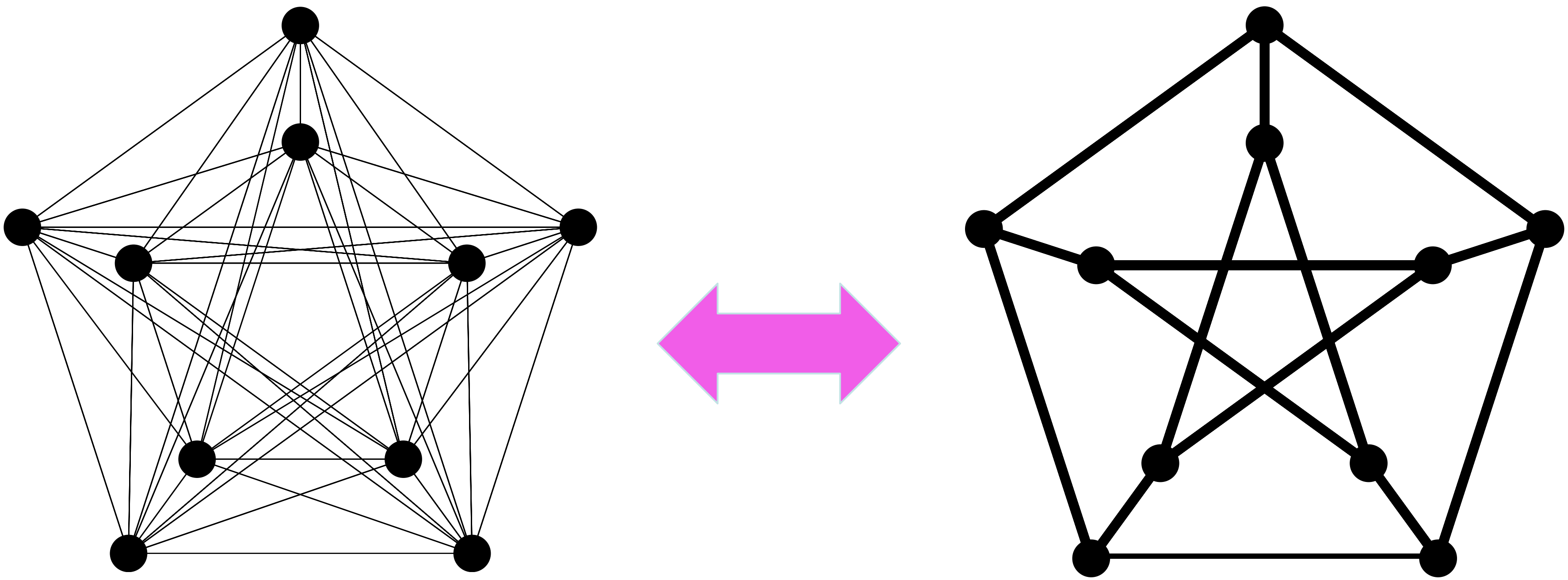
For K_n, the complete graph on n vertices

$\lambda_2, ..., \lambda_n = n$, so for $\begin{matrix} 1^T x = 0 \\ \|x\| = 1 \end{matrix}$ $\quad x^T L_{K_n} x = n$

$$L_{K_n} \approx \frac{n}{d} L_G$$

Good Expander Graphs



$$L_{K_n} \approx \frac{n}{d} L_G$$

Sparse Approximations of Graphs (S-Teng '04)

A graph H is a sparse approximation of G
if H has few edges and $L_H \approx L_G$

few: the number of edges in H is
$O(n)$ or $O(n \log n)$, where $n = |V|$

$L_H \approx_\epsilon L_G$ if $\frac{1}{1+\epsilon} \leq \frac{x^T L_H x}{x^T L_G x} \leq 1 + \epsilon$ for all x

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$$\frac{1}{1+\epsilon} L_G \preccurlyeq L_H \preccurlyeq (1+\epsilon) L_G$$

Where $M \preccurlyeq \widetilde{M}$ if $x^T M x \leq x^T \widetilde{M} x$ for all x

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Why we sparsify graphs

To save memory when storing graphs.

To speed up algorithms:

flow problems in graphs (Benczur-Karger '96)

linear equations in Laplacians (S-Teng '04)

Graph Sparsification Theorems

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

$$L_G \approx_\epsilon L_H \quad \text{and} \quad |F| \leq (2+\epsilon)^2 n/\epsilon^2$$

(Batson-S-Srivastava '09)

Graph Sparsification Theorems

For every $G = (V, E, w)$, there is a $H = (V, F, z)$ s.t.

$$L_G \approx_\epsilon L_H \quad \text{and} \quad |F| \leq (2+\epsilon)^2 n/\epsilon^2$$

(Batson-S-Srivastava ‘09)

By careful random sampling, can quickly get

$$|F| \leq O(n \log n/\epsilon^2)$$

(S-Srivastava ‘08)

Laplacian Matrices

$$x^T L_G x = \sum_{(a,b)\in E} (x(a) - x(b))^2$$

$$L_G = \sum_{(a,b)\in E} L_{a,b}$$

$$L_{1,2} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix}$$

Laplacian Matrices

$$x^T L_G x = \sum_{(a,b)\in E} (x(a) - x(b))^2$$

$$L_G = \sum_{(a,b)\in E} L_{a,b}$$

$$= \sum_{(a,b)\in E} u_{a,b} u_{a,b}^T$$

$$u_{a,b} = \delta_a - \delta_b$$

Laplacian Matrices

$$x^T L_G x = \sum_{(a,b) \in E} (x(a) - x(b))^2$$

$$L_G = \sum_{(a,b) \in E} L_{a,b}$$

$$= \sum_{(a,b) \in E} u_{a,b} u_{a,b}^T$$

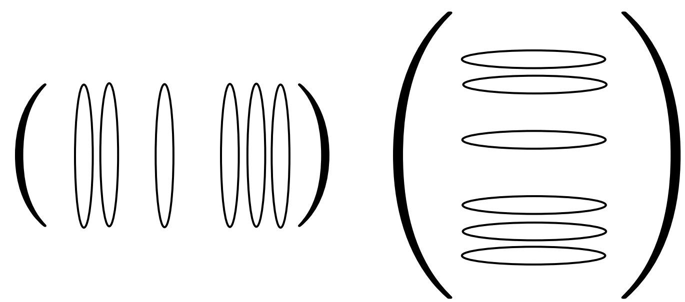
$$u_{a,b} = \delta_a - \delta_b$$

$$= \left(\underset{u_{a,b}}{\Big|} \; U \right) \left(U^T \right)$$

Matrix Sparsification

$$\left(M\right) = \left(\;U\;\right)\left(U^T\right)$$

$$\left(\widetilde{M}\right) =$$



$$\frac{1}{(1+\epsilon)} M \preccurlyeq \widetilde{M} \preccurlyeq (1+\epsilon) M$$

Matrix Sparsification

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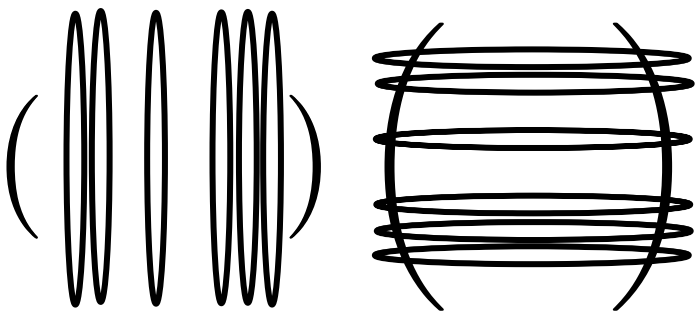
$$(\widetilde{M}) = $$



subset of vectors, scaled up

$$\frac{1}{(1+\epsilon)} M \preccurlyeq \widetilde{M} \preccurlyeq (1+\epsilon) M$$

Matrix Sparsification

$$(M) = (\ U\ )\left(U^T\right)$$

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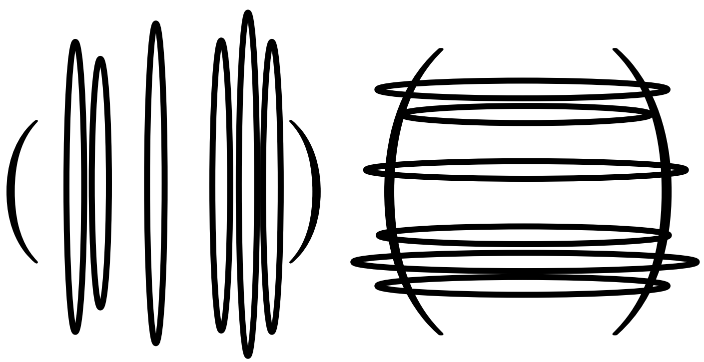


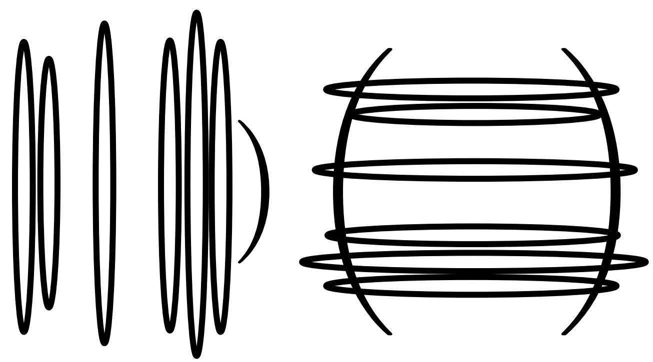
subset of vectors, scaled up

$$\frac{1}{(1+\epsilon)} M \preccurlyeq \widetilde{M} \preccurlyeq (1+\epsilon) M$$

Matrix Sparsification

$$(M) = (\ U\ )\left(U^T\right) = \sum_i u_i u_i^T$$

$$(\widetilde{M}) = (\quad)(\quad) = \sum_i s_i u_i u_i^T$$



most $s_i = 0$

$$\frac{1}{(1+\epsilon)} M \preccurlyeq \widetilde{M} \preccurlyeq (1+\epsilon) M$$

Simplification of Matrix Sparsification

$$\frac{1}{(1+\epsilon)} M \preccurlyeq \widetilde{M} \preccurlyeq (1+\epsilon) M$$

is equivalent to

$$\frac{1}{(1+\epsilon)} I \preccurlyeq M^{-1/2} \widetilde{M} M^{-1/2} \preccurlyeq (1+\epsilon) I$$

Simplification of Matrix Sparsification

$$\frac{1}{(1+\epsilon)} I \preccurlyeq M^{-1/2} \widetilde{M} M^{-1/2} \preccurlyeq (1+\epsilon) I$$

Set $v_i = M^{-1/2} u_i$ $\qquad$ $\sum_i v_i v_i^T = I$

We need $\qquad$ $\sum_i s_i v_i v_i^T \approx_\epsilon I$

Simplification of Matrix Sparsification

$$\frac{1}{(1+\epsilon)} I \preccurlyeq M^{-1/2} \widetilde{M} M^{-1/2} \preccurlyeq (1+\epsilon) I$$

Set $v_i = M^{-1/2} u_i$

$$\sum_i v_i v_i^T = I$$

"Decomposition of the identity"
"Parseval frame"
"Isotropic Position"

$$\sum_i (v_i^T t)^2 = \|t\|^2$$

Matrix Sparsification by Sampling

(Rudelson '99, Ahlswede-Winter '02, Tropp '11)

For $v_1, \ldots, v_m \in \mathbb{R}^n$ with $\sum_i v_i v_i^T = I$

Choose v_i with probability $p_i \sim \|v_i\|^2$

If choose v_i, set $s_i = 1/p_i$

$$s_i = \begin{cases} 1/p_i & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases}$$

$$\mathbb{E}\left[\sum_i s_i v_i v_i^T\right] = \sum_i v_i v_i^T$$

Matrix Sparsification by Sampling

(Rudelson '99, Ahlswede-Winter '02, Tropp '11)

For $v_1, \ldots, v_m \in \mathbb{R}^n$ with $\sum_i v_i v_i^T = I$

Choose v_i with probability $p_i \sim \|v_i\|^2$

If choose v_i, set $s_i = 1/p_i$ (effective conductance)

$$s_i = \begin{cases} 1/p_i & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases}$$

$$\mathbb{E}\left[\sum_i s_i v_i v_i^T \right] = \sum_i v_i v_i^T$$

Matrix Sparsification by Sampling

(Rudelson '99, Ahlswede-Winter '02, Tropp '11)

For $v_1, ..., v_m \in \mathbb{R}^n$ with $\sum_i v_i v_i^T = I$

Choose v_i with probability $p_i = C(\log n) \|v_i\|^2 / \epsilon^2$

If choose v_i, set $s_i = 1/p_i$

$$s_i = \begin{cases} 1/p_i & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases}$$

$$\mathbb{E}\left[\sum_i s_i v_i v_i^T \right] = \sum_i v_i v_i^T$$

Matrix Sparsification by Sampling

(Rudelson '99, Ahlswede-Winter '02, Tropp '11)

For $v_1, \ldots, v_m \in \mathbb{R}^n$ with $\sum_i v_i v_i^T = I$

Choose v_i with probability $p_i = C(\log n) \left\| v_i \right\|^2 / \epsilon^2$

If choose v_i, set $s_i = 1/p_i$

With high probability, choose $O(n \log n / \epsilon^2)$ vectors

and $\sum_i s_i v_i v_i^T \approx_\epsilon I$

Optimal (?) Matrix Sparsification

(Batson-S-Srivastava ‘09)

For $v_1, ..., v_m \in \mathbb{R}^n$ with $\sum_i v_i v_i^T = I$

Can choose $(2+\epsilon)^2 n/\epsilon^2$ vectors
and nonzero values for the s_i so that

$$\sum_i s_i v_i v_i^T \approx_\epsilon I$$

Optimal (?) Matrix Sparsification

(Batson-S-Srivastava '09)

For $v_1, ..., v_m \in \mathbb{R}^n$ with $\sum_i v_i v_i^T = I$

Can choose $(2+\epsilon)^2 n/\epsilon^2$ vectors
and nonzero values for the s_i so that

$$\sum_i s_i v_i v_i^T \approx_\epsilon I$$

What are the s_i !?

Optimal (?) Matrix Sparsification

(Batson-S-Srivastava '09)

For $v_1, \ldots, v_m \in \mathbb{R}^n$ with $\sum_i v_i v_i^T = I$

Can choose $(2+\epsilon)^2 n/\epsilon^2$ vectors
and nonzero values for the s_i so that

$$\sum_i s_i v_i v_i^T \approx_\epsilon I$$

$$s_i \sim 1/\|v_i\|^2$$ ⁉

The Kadison-Singer Problem ‘59

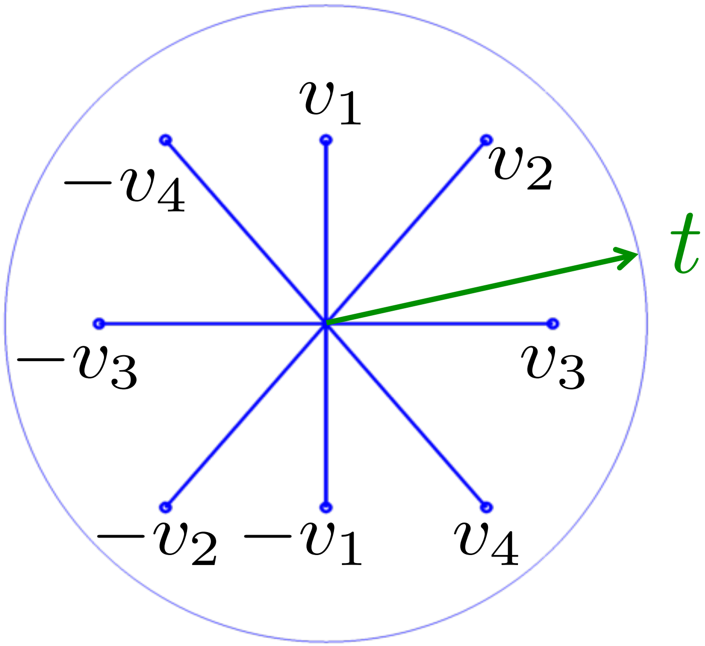
Equivalent to:

- Anderson’s Paving Conjectures (‘79, ‘81)
- Bourgain-Tzafriri Conjecture (‘91)
- Feichtinger Conjecture (‘05)
- Many others

Implied by:

- Weaver’s KS_2 conjecture (‘04)

Weaver's Conjecture: Isotropic vectors

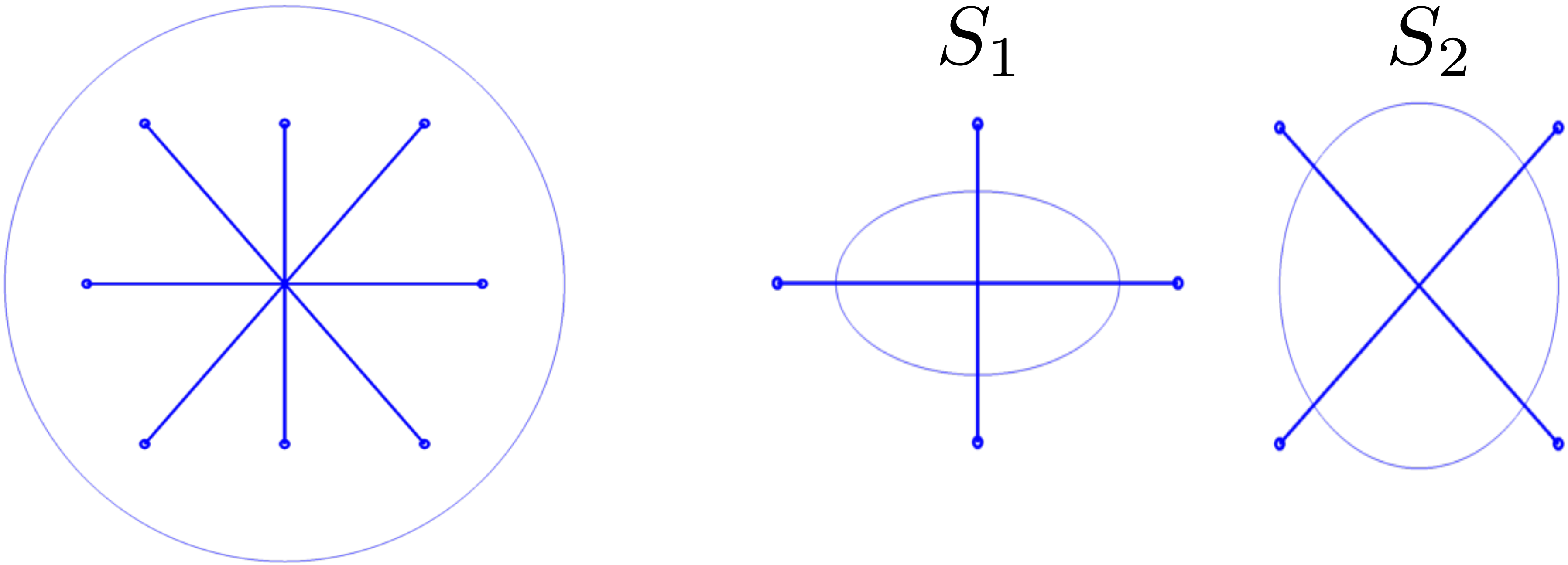


$$\sum_i v_i v_i^T = I$$

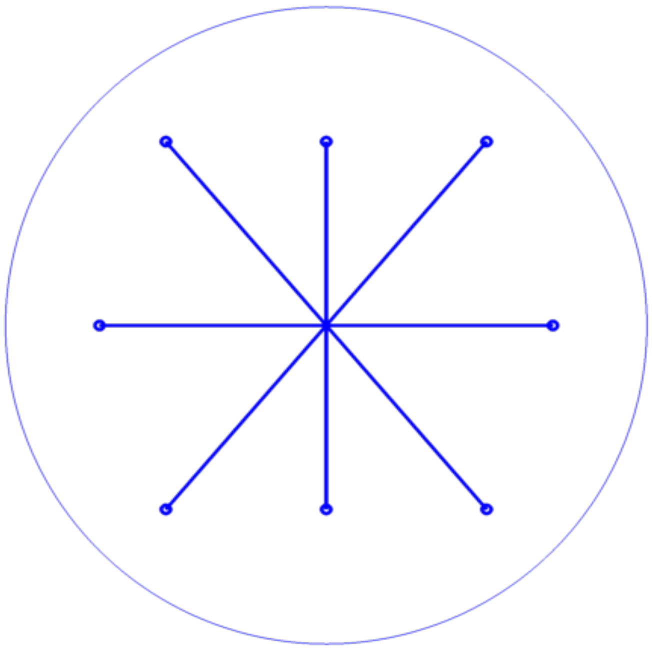
for every unit vector t

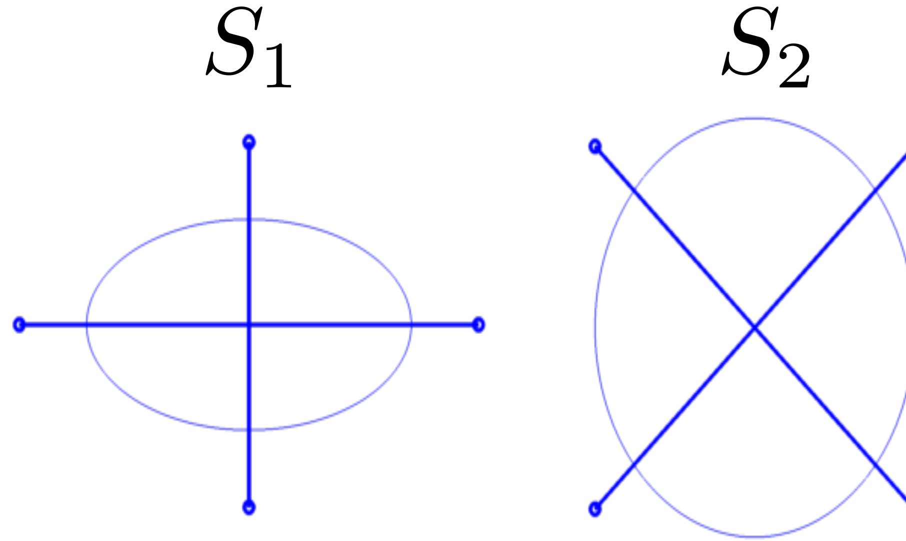
$$\sum_i (v_i^T t)^2 = 1$$

Partition into approximately ½-Isotropic Sets



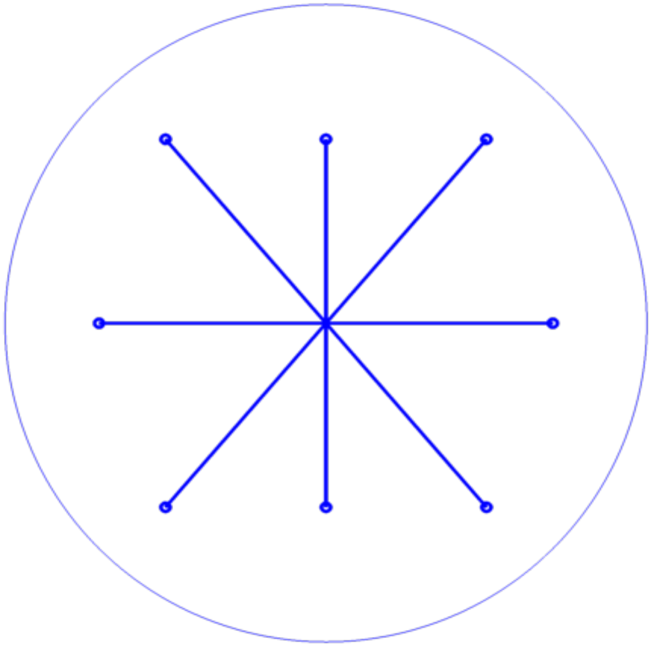
Partition into approximately ½-Isotropic Sets

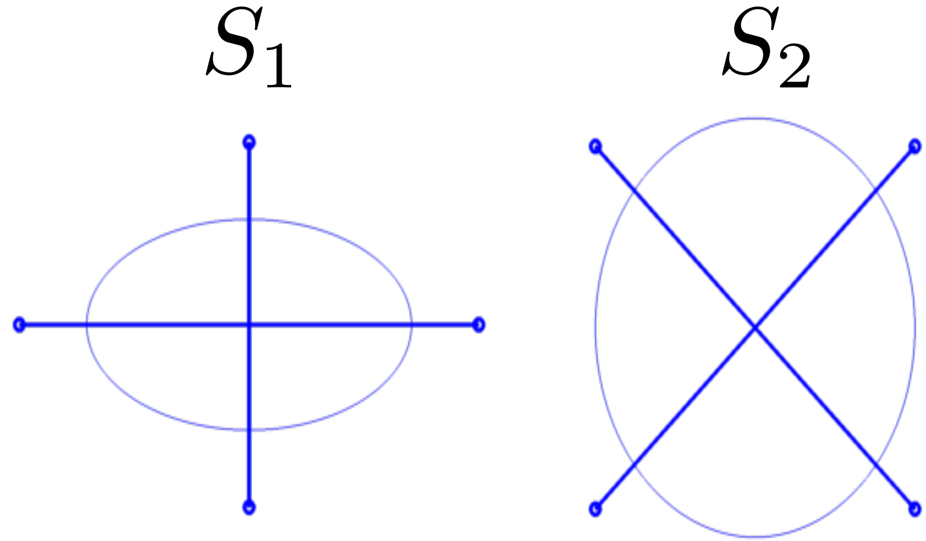




$$1/4 \le \sum_{i \in S_j} (v_i^T t)^2 \le 3/4$$

Partition into approximately ½-Isotropic Sets

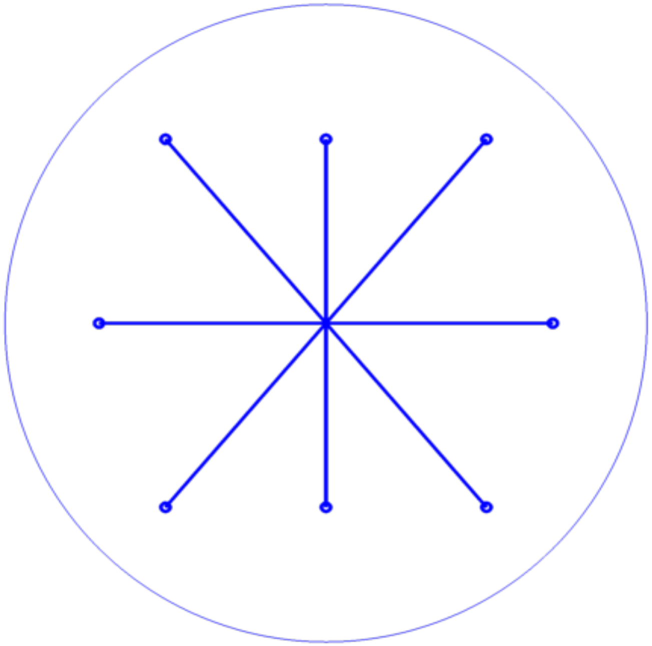




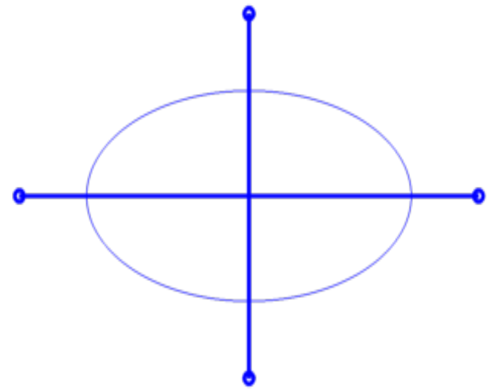
$$1/4 \leq \sum_{i \in S_j} (v_i^T t)^2 \leq 3/4$$

$$1/4 \leq \text{eigs}(\sum_{i \in S_j} v_i v_i^T) \leq 3/4$$

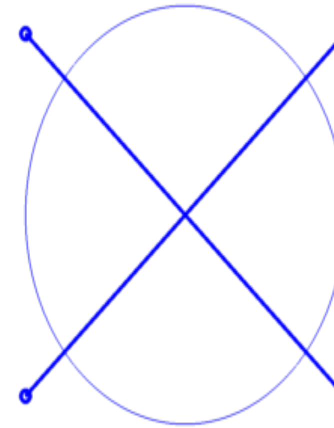
Partition into approximately ½-Isotropic Sets



S_1



S_2



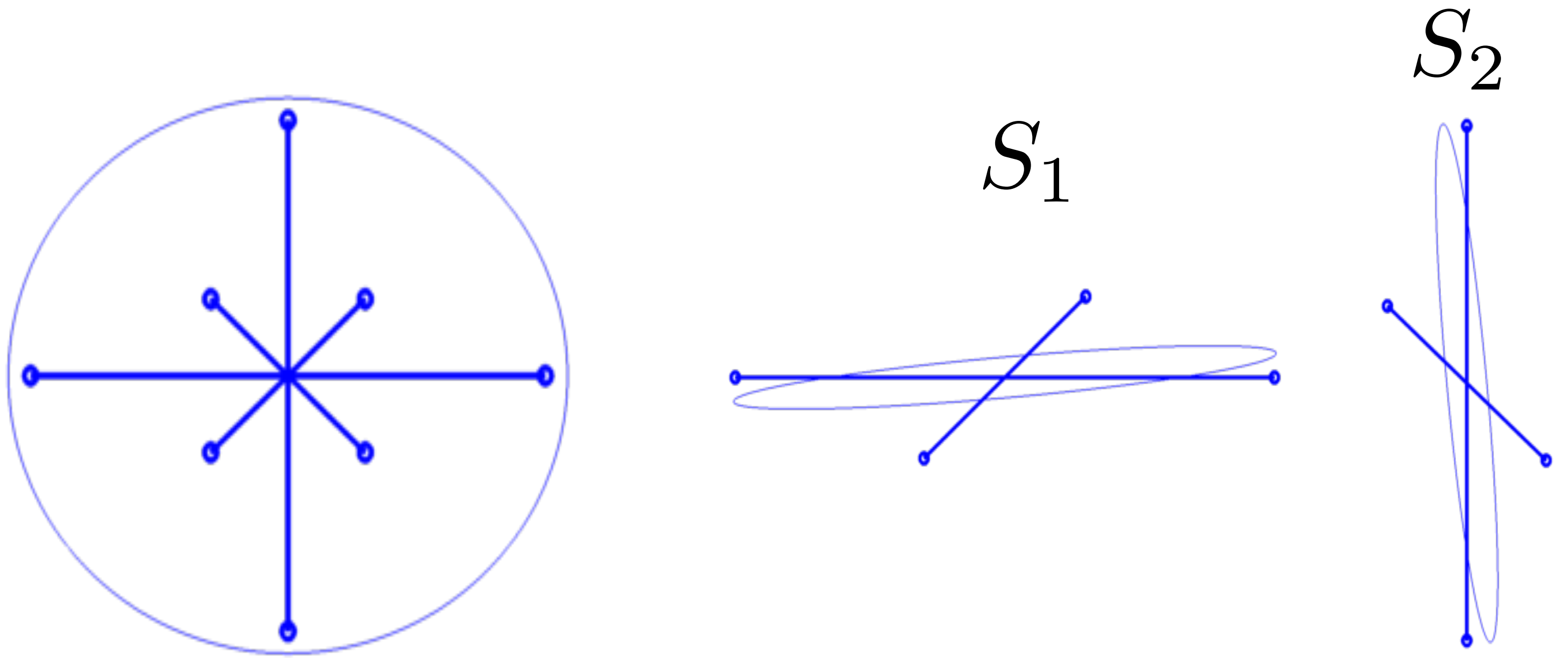
$$1/4 \leq \sum_{i \in S_j} (v_i^T t)^2 \leq 3/4$$

$$1/4 \leq \text{eigs}(\sum_{i \in S_j} v_i v_i^T) \leq 3/4$$

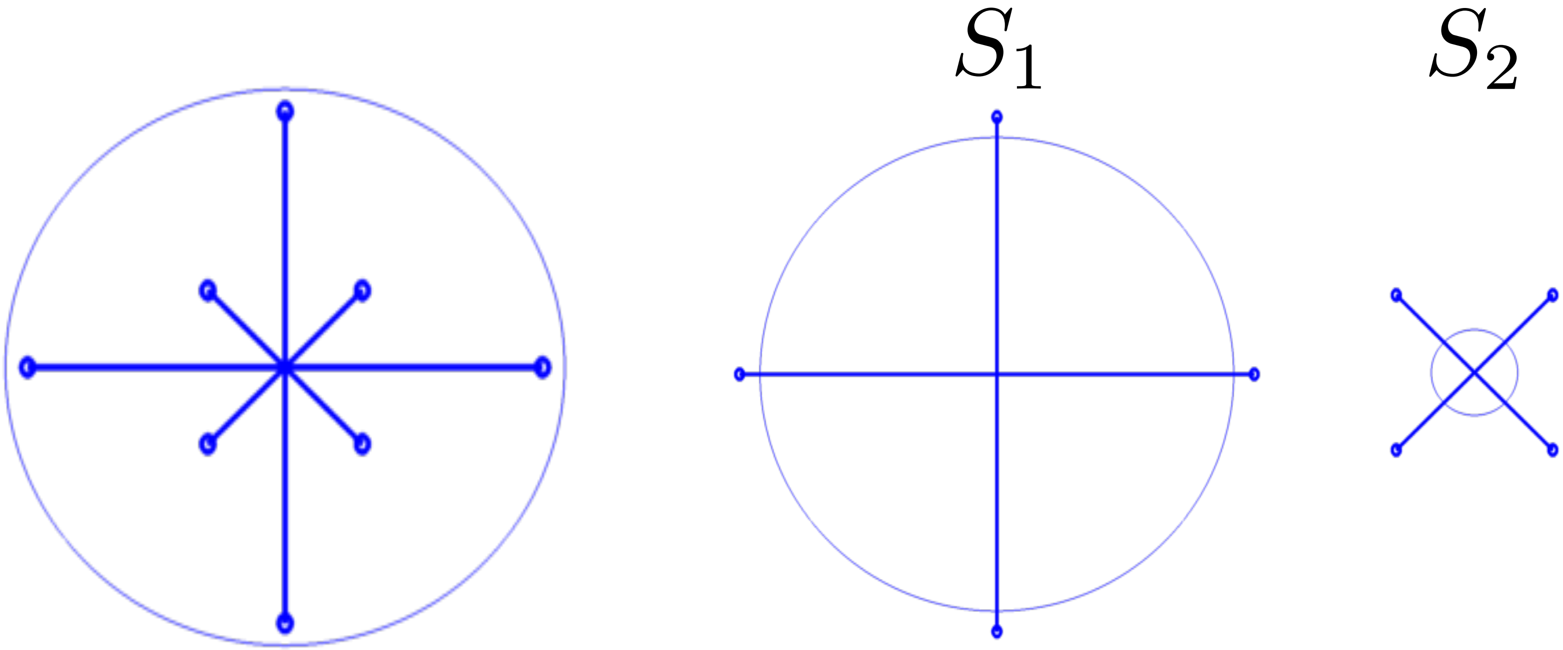
$$\Longleftrightarrow \quad \text{eigs}(\sum_{i \in S_j} v_i v_i^T) \leq 3/4$$

because $$\sum_{i \in S_1} v_i v_i^T = I - \sum_{i \in S_2} v_i v_i^T$$

Big vectors make this difficult



Big vectors make this difficult



Weaver's Conjecture KS_2

There exist positive constants α and ϵ so that

if all $\|v_i\|^2 \leq \alpha$ and $\sum v_i v_i^T = I$

then exists a partition into S_1 and S_2 with

$$\text{eigs}(\sum_{i \in S_j} v_i v_i^T) \leq 1 - \epsilon$$

Theorem (Marcus-S-Srivastava '15)

For all $\alpha > 0$

if all $\|v_i\|^2 \leq \alpha$ and $\sum v_i v_i^T = I$

then exists a partition into S_1 and S_2 with

$$\text{eigs}\left(\sum_{i \in S_j} v_i v_i^T\right) \leq \frac{1}{2} + 3\alpha$$

We want

$$\text{eigs}\begin{pmatrix} \sum_{i\in S_1} v_i v_i^T & 0 \\ 0 & \sum_{i\in S_2} v_i v_i^T \end{pmatrix} \leq \frac{1}{2} + 3\alpha$$

We want

$$\mathrm{roots}\left(\mathrm{poly}\begin{pmatrix} \sum_{i \in S_1} v_i v_i^T & 0 \\ 0 & \sum_{i \in S_2} v_i v_i^T \end{pmatrix}\right) \leq \frac{1}{2} + 3\alpha$$

We want

$$\text{roots}\left(\text{poly}\begin{pmatrix}\sum_{i\in S_1} v_i v_i^T & 0 \\ 0 & \sum_{i\in S_2} v_i v_i^T\end{pmatrix}\right) \leq \frac{1}{2} + 3\alpha$$

Consider expected polynomial of a random partition.

Proof Outline

1. Prove expected characteristic polynomial
   has real roots

2. Prove its largest root is at most $1/2 + 3\alpha$

3. Prove is an interlacing family, so
   exists a partition whose polynomial
   has largest root at most $1/2 + 3\alpha$

Interlacing

Polynomial $p(x) = \prod_{i=1}^{d} (x - \alpha_i)$

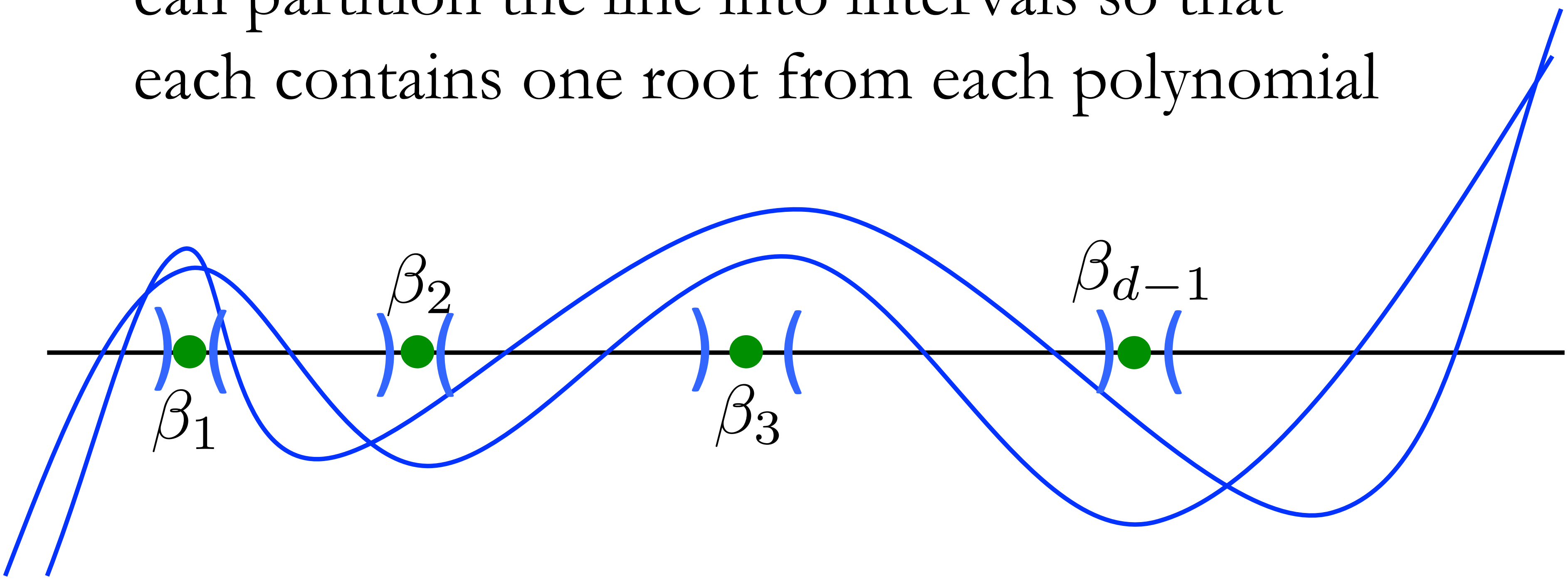
interlaces $q(x) = \prod_{i=1}^{d-1} (x - \beta_i)$

if $\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \alpha_{d-1} \leq \beta_{d-1} \leq \alpha_d$

Example: $q(x) = \frac{d}{dx} p(x)$

Common Interlacing

$p_1(x)$ and $p_2(x)$ have a common interlacing if can partition the line into intervals so that each contains one root from each polynomial

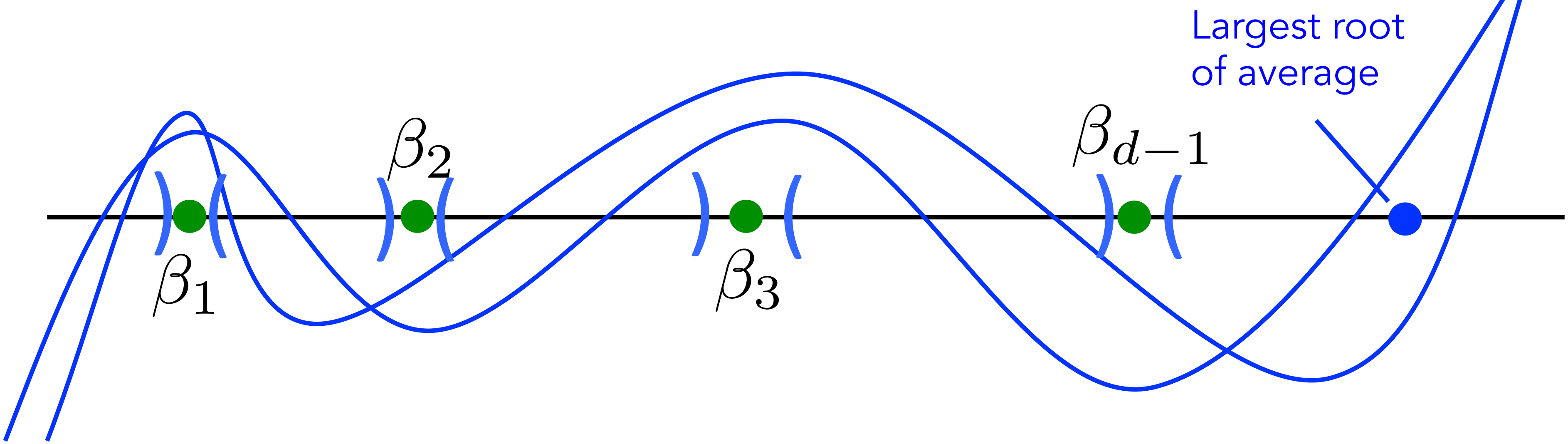


Common Interlacing

If p_1 and p_2 have a common interlacing,

$$\text{max-root}\,(p_i) \leq \text{max-root}\,(\mathbb{E}_i\,[\;p_i\;])$$

for some i.



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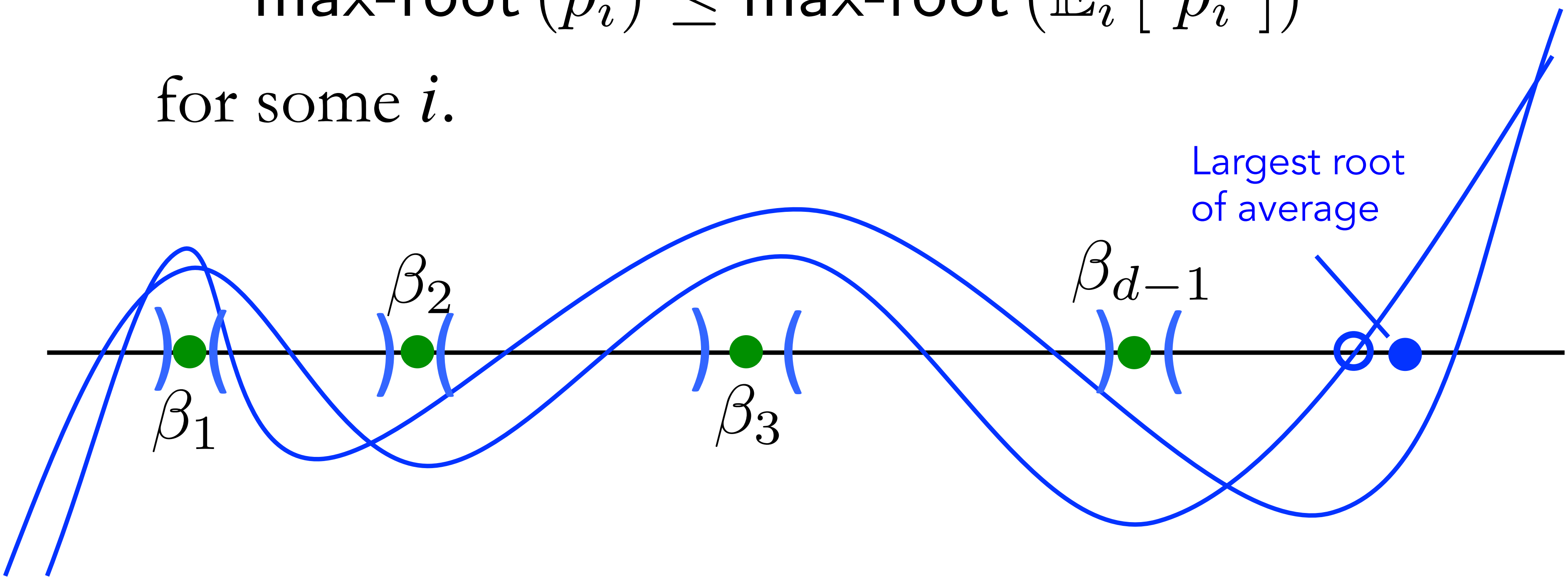
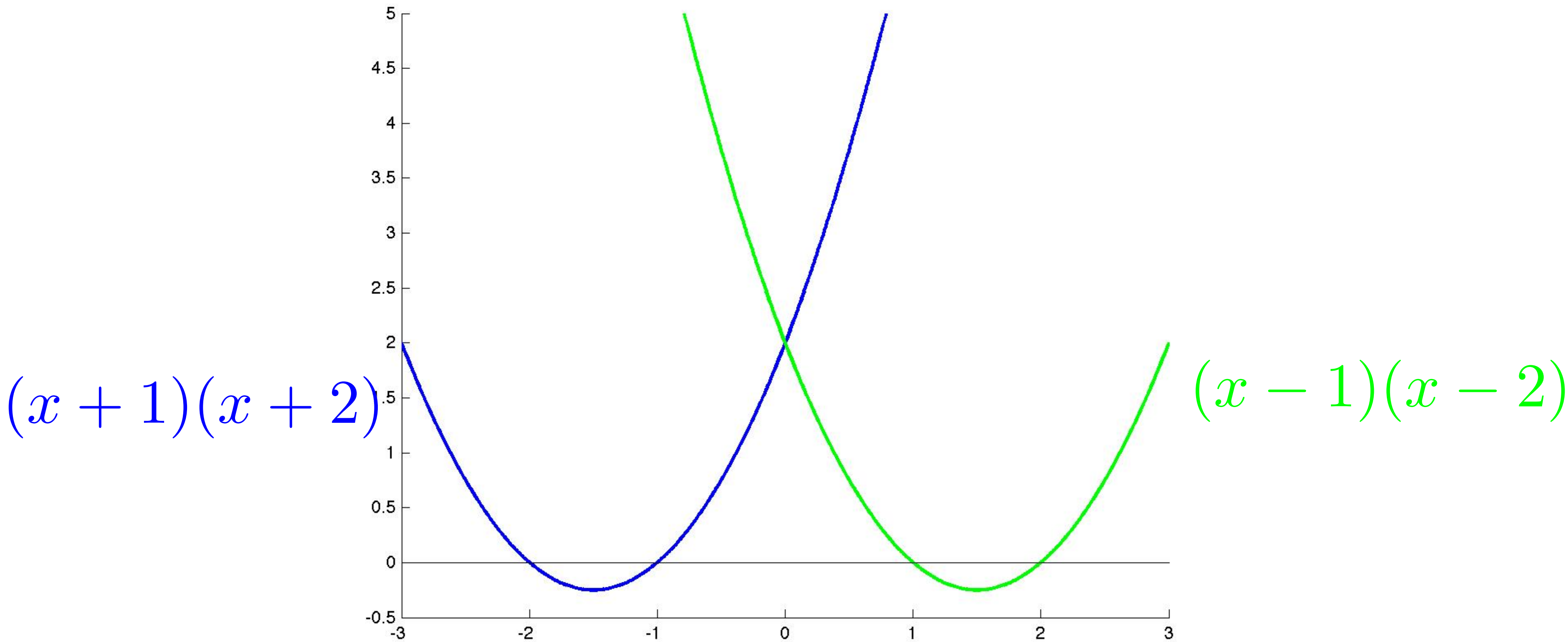
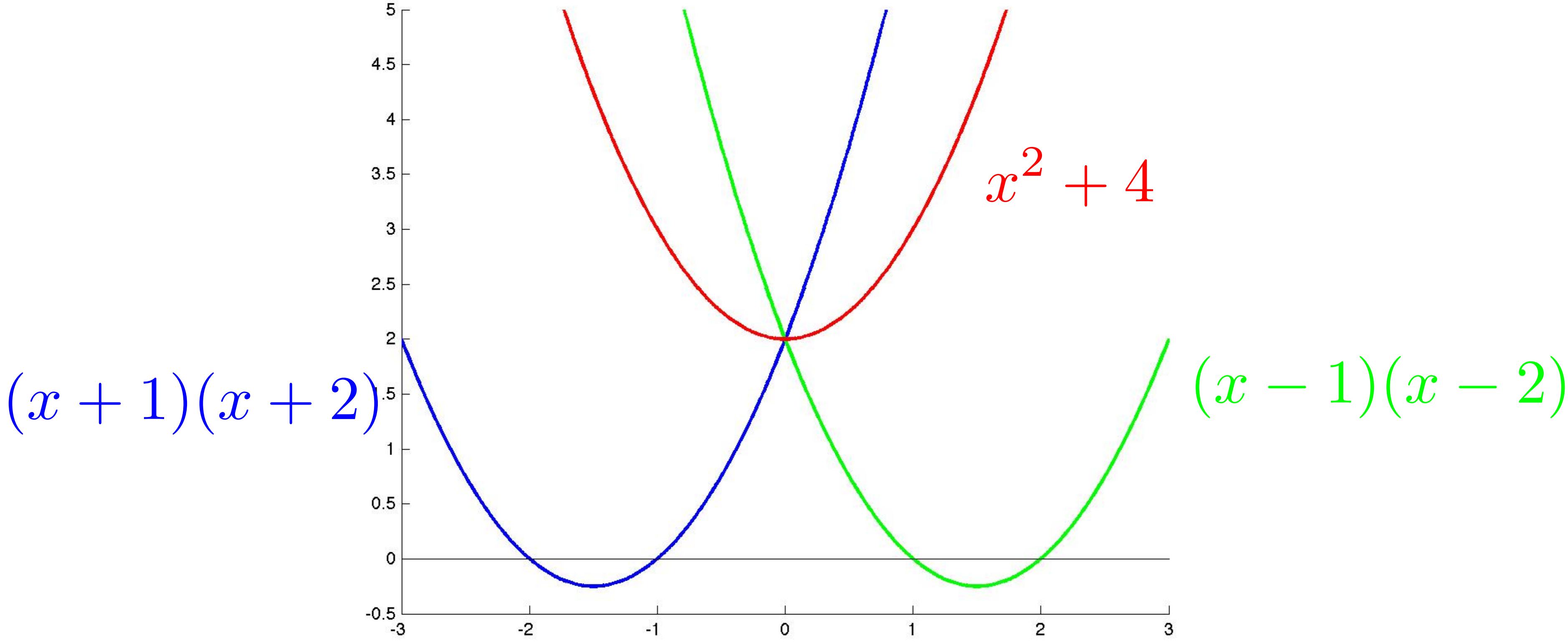


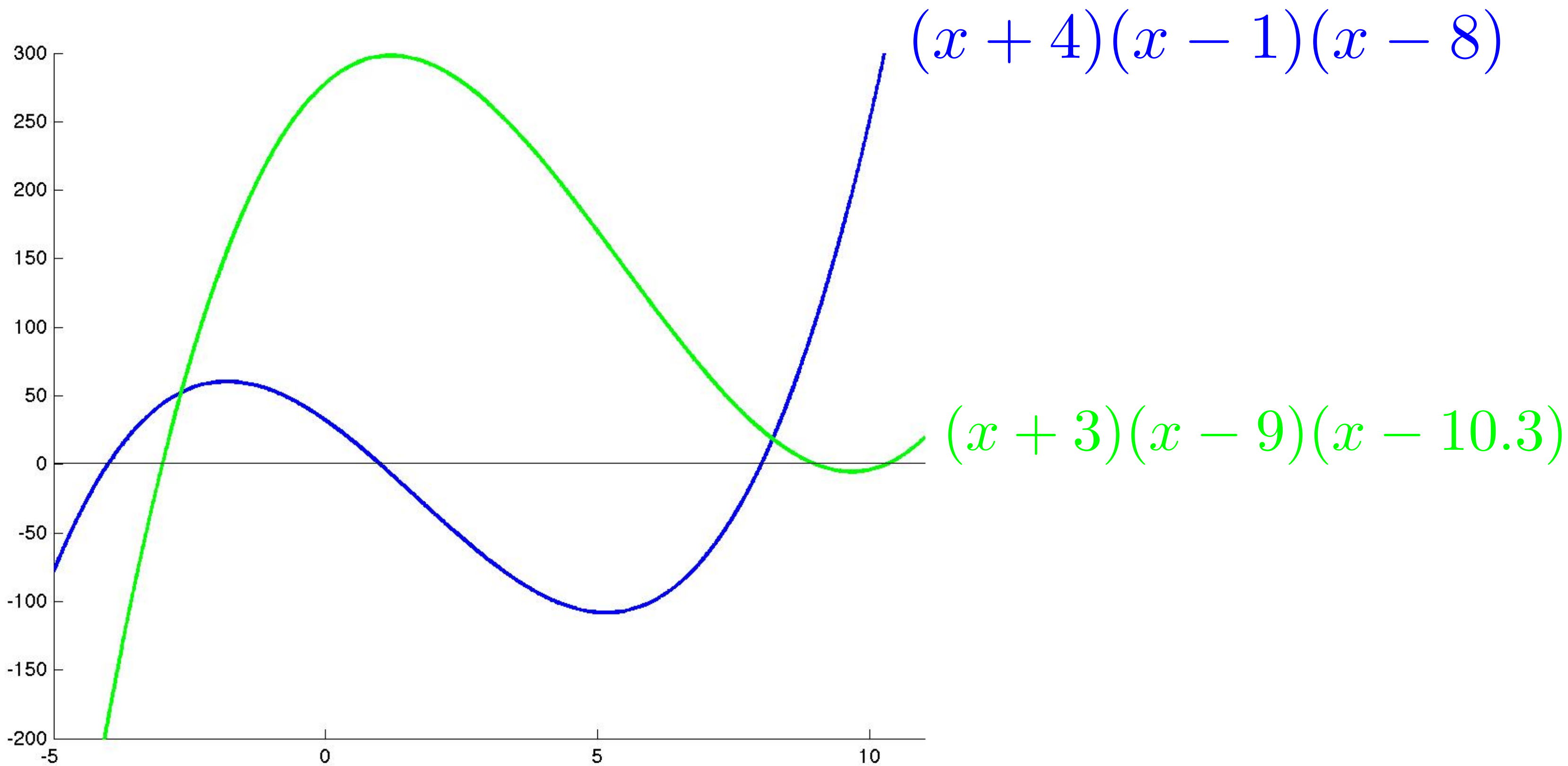
Without a common interlacing



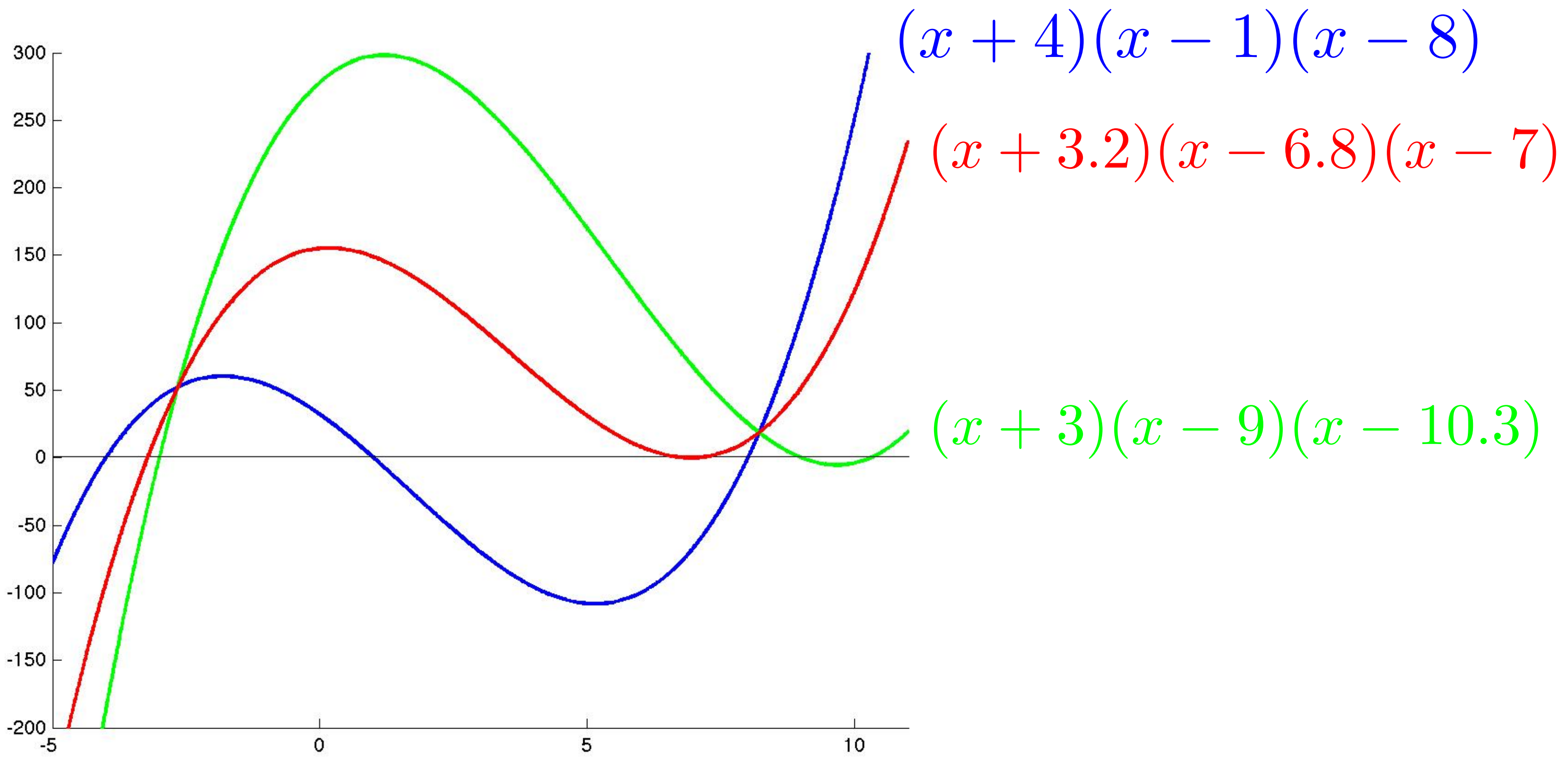
Without a common interlacing



Without a common interlacing



Without a common interlacing



Common Interlacing

If p_1 and p_2 have a common interlacing,

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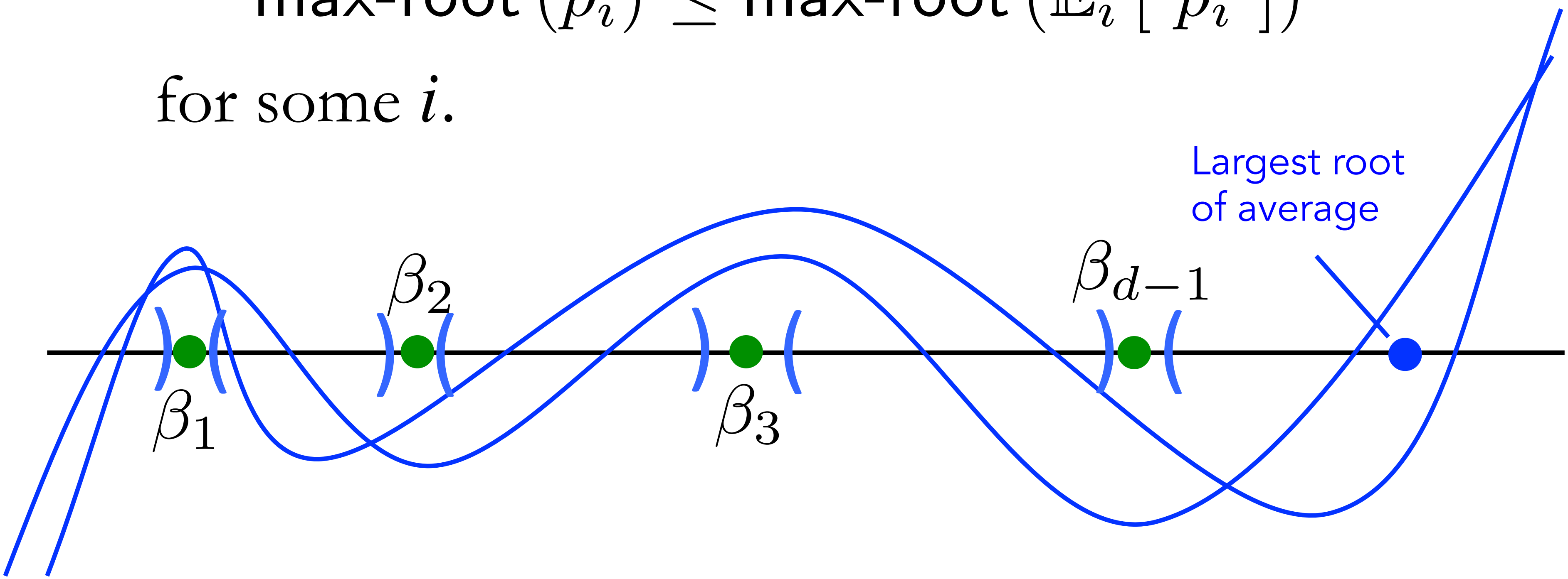
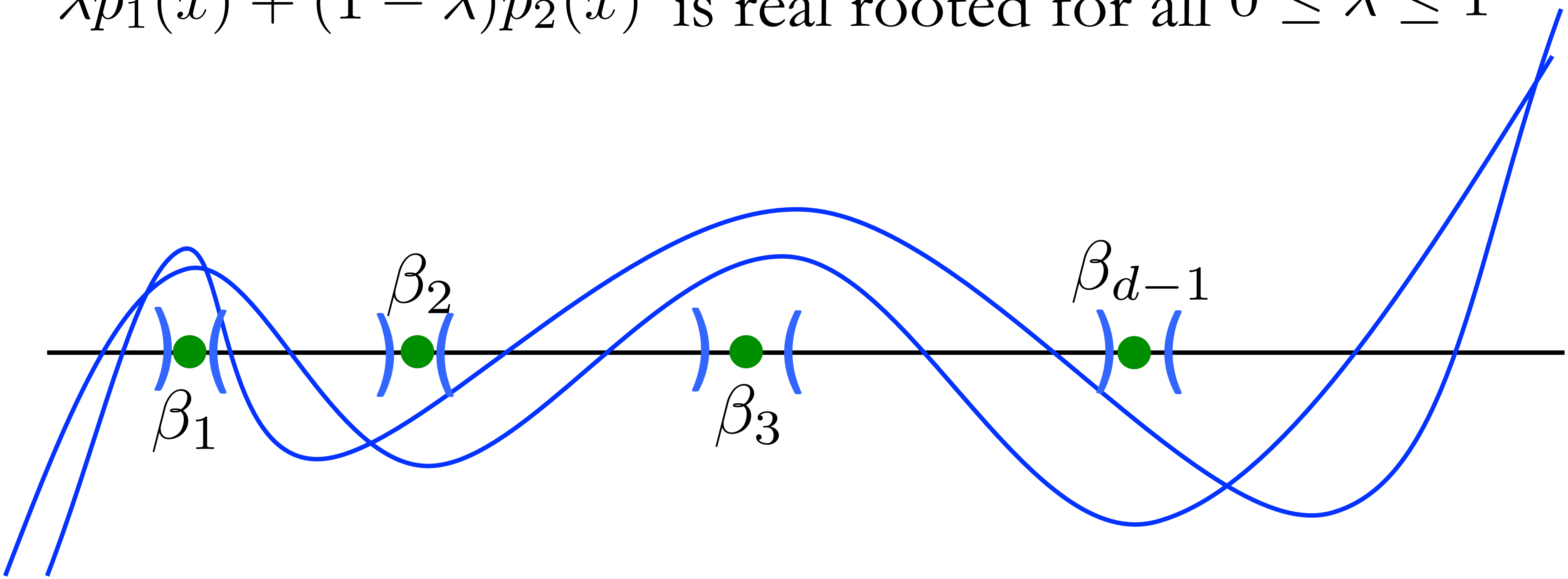
for some i.



Common Interlacing

$p_1(x)$ and $p_2(x)$ have a common interlacing iff

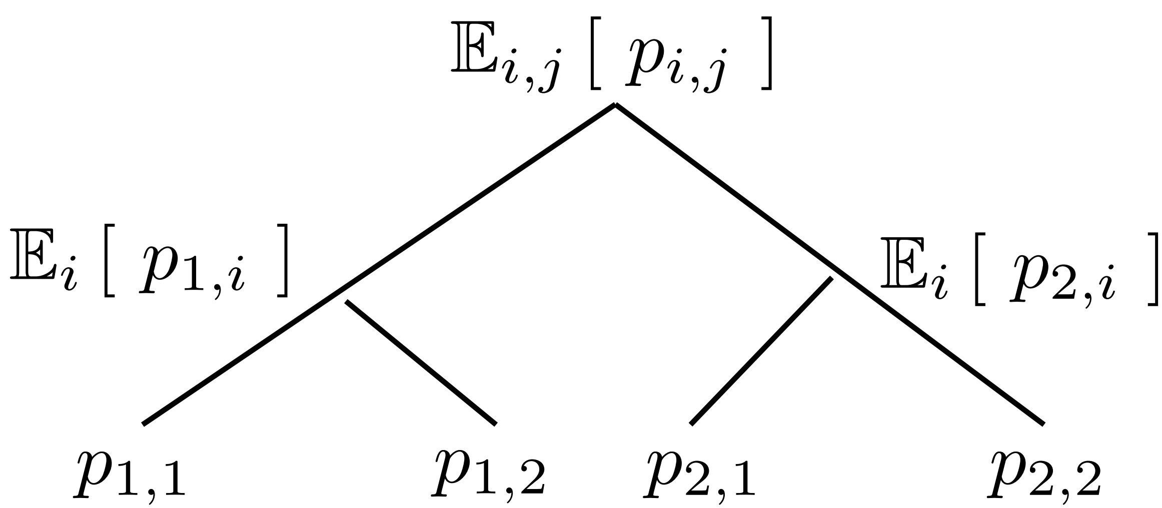
$\lambda p_1(x) + (1-\lambda)p_2(x)$ is real rooted for all $0 \le \lambda \le 1$



Interlacing Family of Polynomials

$\{p_\sigma\}_{\sigma \in \{1,2\}^n}$ is an interlacing family

if its members can be placed on the leaves of a tree so that when every node is labeled with the average of leaves below, siblings have common interlacings



Interlacing Family of Polynomials

$$\{p_\sigma\}_{\sigma \in \{1,2\}^n} \quad \text{is an interlacing family}$$

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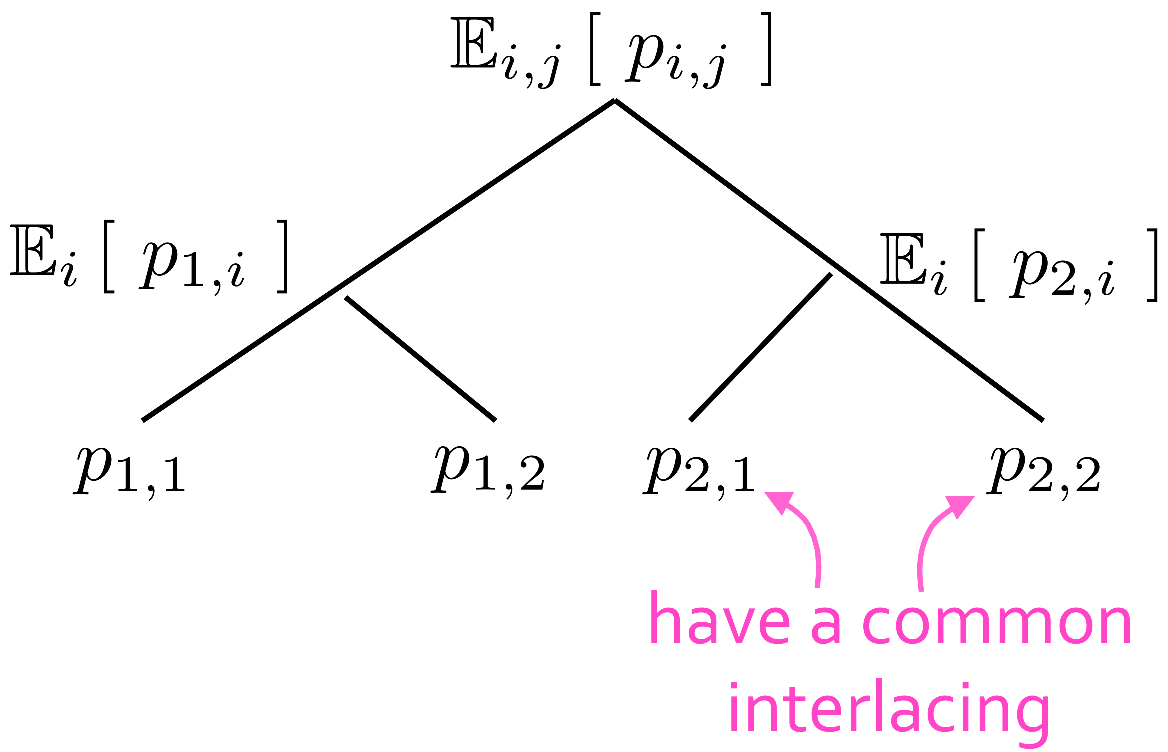

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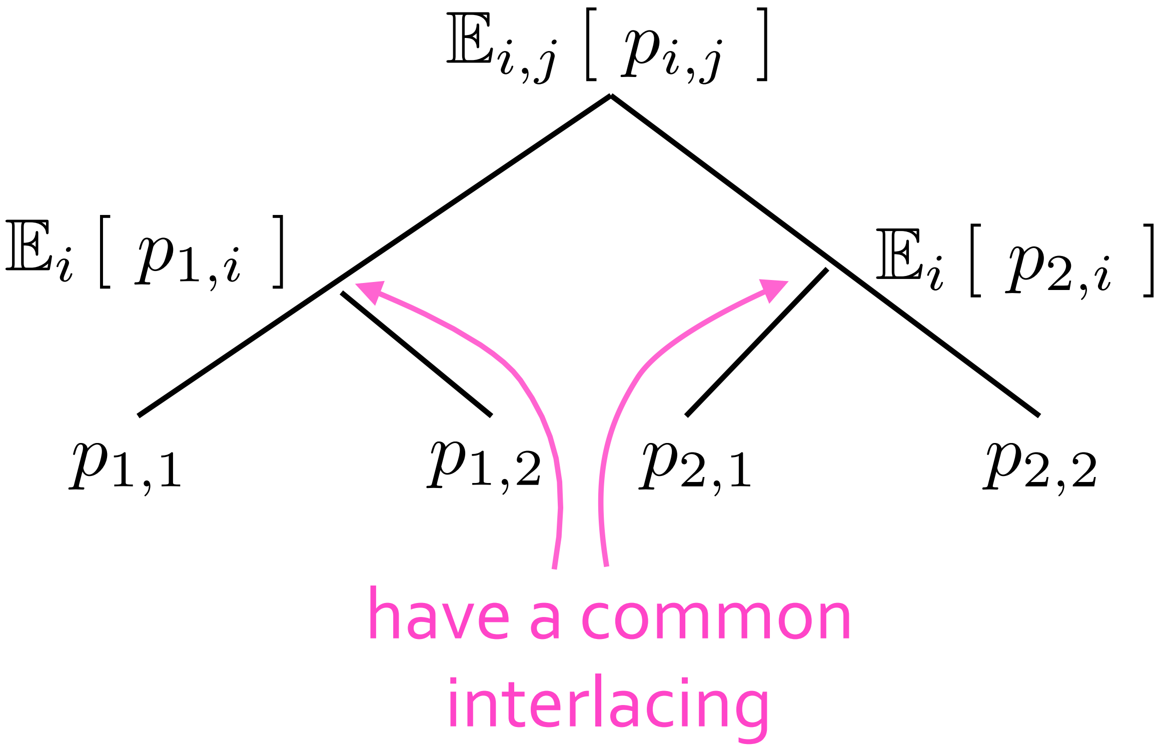
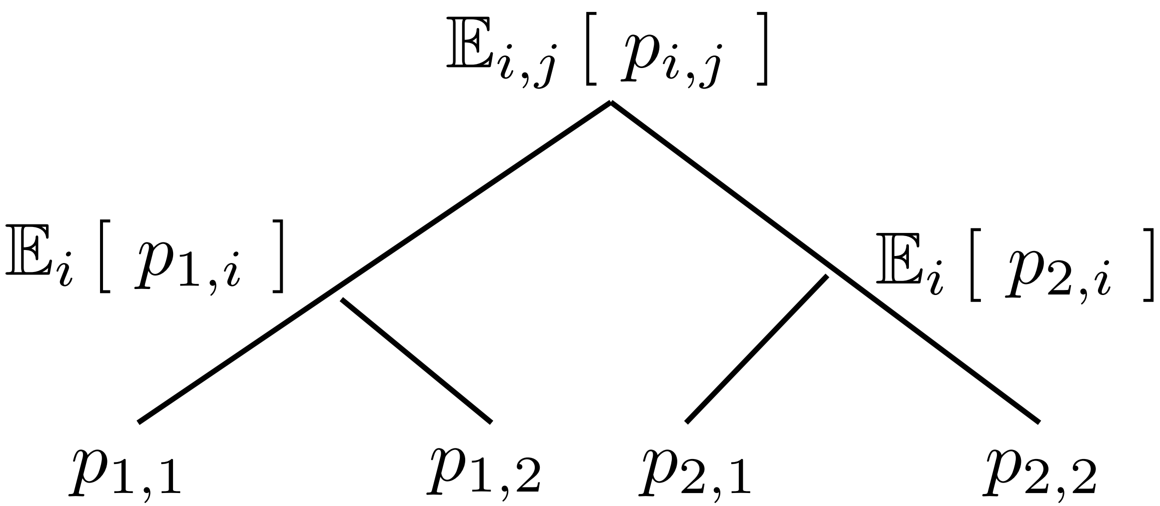


Interlacing Family of Polynomials

Theorem:

There is a σ so that

$$\text{max-root}(p_\sigma) \leq \text{max-root}(\mathbb{E}_\sigma p_\sigma)$$



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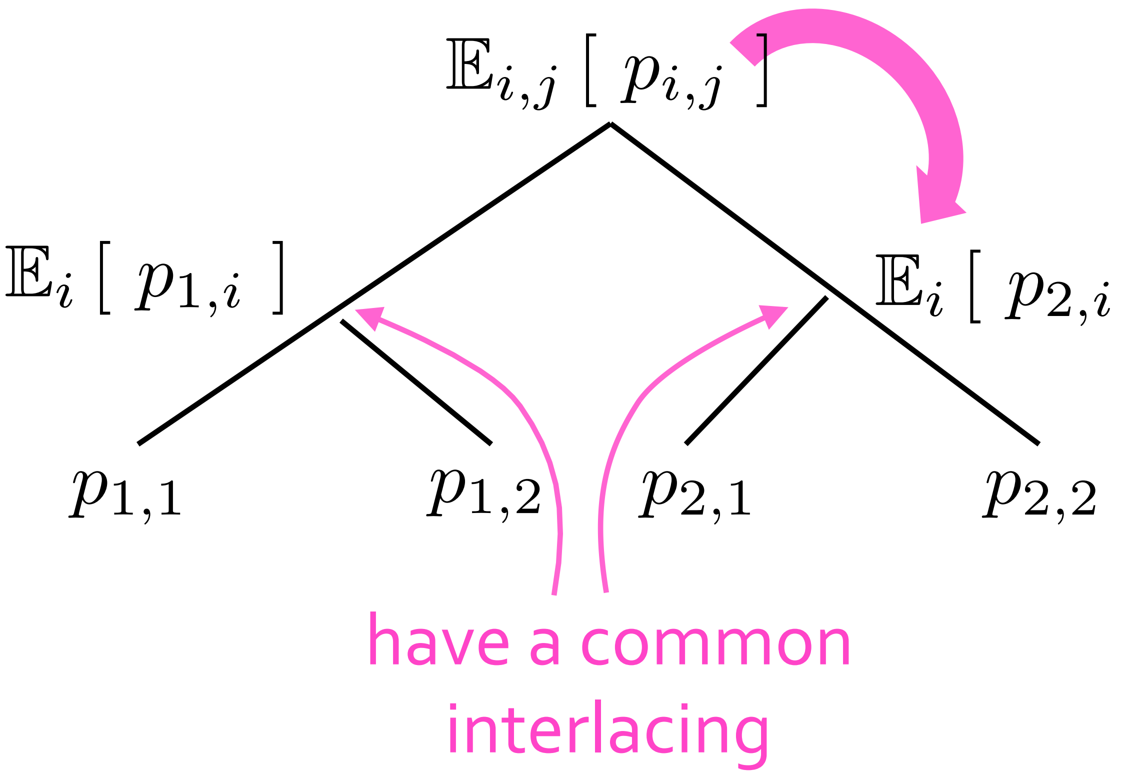

Interlacing Family of Polynomials

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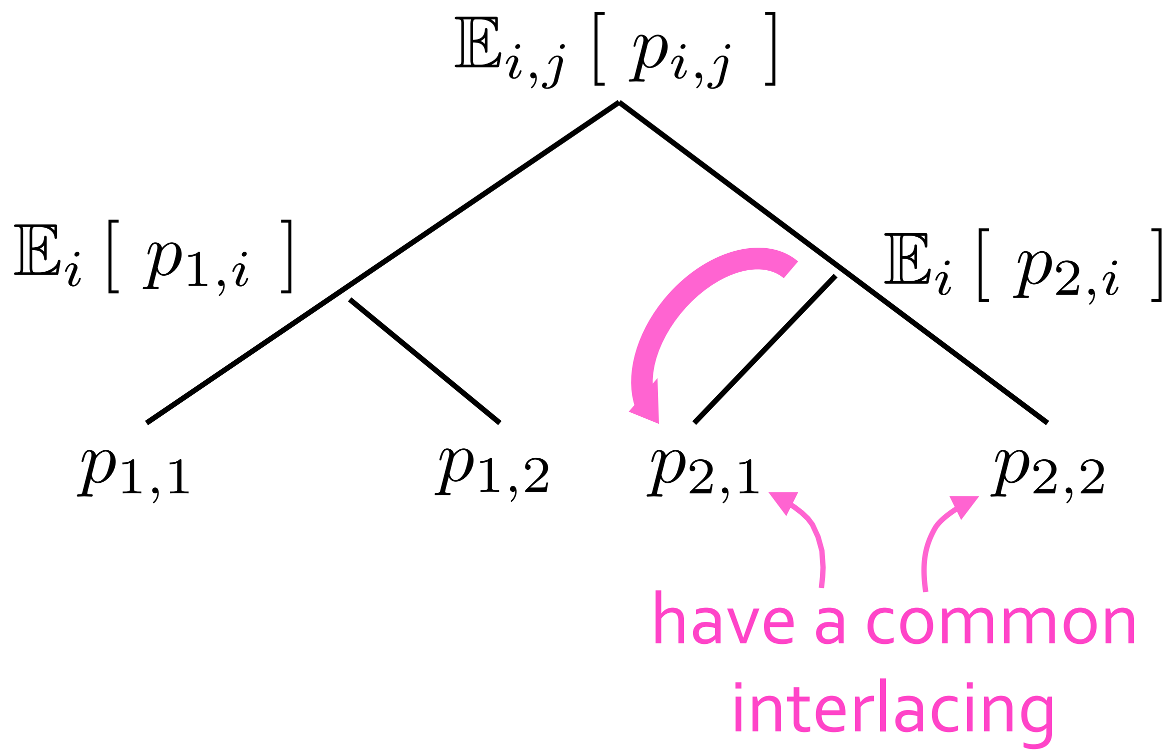

Our family is interlacing

$$\mathbb{E}_{S_1,S_2}\left[\text{ poly}\begin{pmatrix} \sum_{i\in S_1} v_i v_i^T & 0 \\ 0 & \sum_{i\in S_2} v_i v_i^T \end{pmatrix}\right]$$

Form other polynomials in the tree
by fixing the choices of where some vectors go

Summary

1. Prove expected characteristic polynomial
   has real roots

2. Prove its largest root is at most $1/2 + 3\alpha$

3. Prove is an interlacing family, so
   exists a partition whose polynomial
   has largest root at most $1/2 + 3\alpha$

To learn more about Laplacians, see

My class notes from

“Spectral Graph Theory” and “Graphs and Networks”

My web page on

Laplacian linear equations, sparsification, etc.

To learn more about Kadison-Singer

Papers in Annals of Mathematics and survey from ICM.

Available on arXiv and my web page