Linear Programming & Convex Sets.

An LP has variables $x \in \mathbb{R}^d$,

an objective function $\max c^T x$, specified by $c \in \mathbb{R}^d$

subject to m constraints $a_i^T x \le b_i$,

where $a_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$, for $1 \le i \le m$.

We typically collect the a_i into an m-by-d matrix, A, and the b_i into an m-vector, b, and write

$Ax \le b$, where this means the inequality holds in every coordinate:

$x \le y$ iff for all i $x(i) \le y(i)$.

We will see many forms of linear programs.

If want $a^T x \ge b$, write it as $(-a)^T x \le -b$.

If want $a^T x = b$, write it as $a^T x \le b$ and $(-a)^T x \le -b$

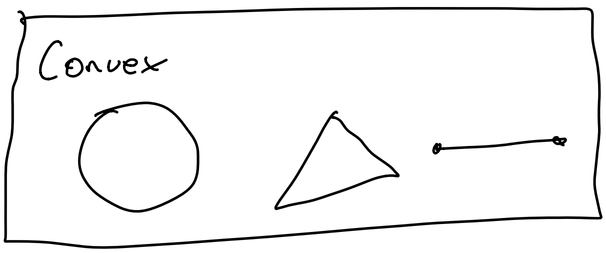
The set $\{x : Ax \le b\}$ is a polyhedron.

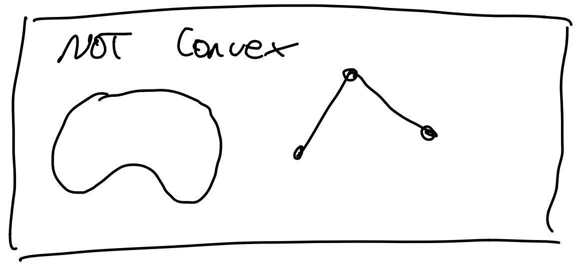
In particular, it is convex.

We will spend most of this lecture understanding convex sets.

<u>Def</u> $C \subseteq \mathbb{R}^n$ is convex if for all $x, y \in C$ and all $0 \le t \le 1$, $tx + (1-t)y \in C$.

That is, the line segment from x to y is in C.





<u>Examples</u>

A subspace : $\{x : Ax = 0\}$

$$Ax = 0 \text{ and } Ay = 0 \Rightarrow A(tx + (1-t)y) = 0.$$

An affine space : $\{x : Ax = b\}$

$$Ax = b \text{ and } Ay = b \Rightarrow$$
$$A(tx + (1-t)y) = tAx + (1-t)Ay = tb + (1-t)b = b.$$

A line segment : $\{x : a \le x \le b\}$, $a, b \in \mathbb{R}$

An open line segment : $\{x : a < x < b\}$, $a, b \in \mathbb{R}$

Positive reals : $x > 0$

A halfspace : $x \in \mathbb{R}^d$ s.t. $a^T x \le b$

$$a^T(tx + (1-t)y) = ta^T x + (1-t)a^T y$$
$$\le tb + (1-t)b = b$$

An open halfspace : $x \in \mathbb{R}^d$ s.t. $a^T x < b$

A norm ball: $\{x : \|x\| \le 1\}$, for any norm $\|\cdot\|$

Let $B(x,r)$ be the ball of radius r around x

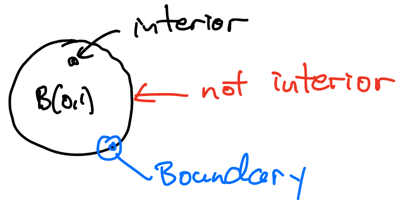
$$= \{y : \|x-y\|_2 \le r\}$$

Some topology

For $S \subseteq \mathbb{R}^d$, x is in the interior of S

if $\exists\, \varepsilon > 0$ s.t. $B(x,\varepsilon) \subseteq S$

Ex:



x is on the boundary of S if $\forall \varepsilon > 0$,

$B(x,\varepsilon)$ contains points in S and not in S

If $S = \{x : 0 < x \le 1\}$, The boundary $= \{0,1\}$

$1 \in S$ but $0 \notin S$

S is closed if boundary$(S) \subseteq S$

S is open if $S = \text{interior}(S)$

e.g. $\{x : 0 < x < 1\}$ $\forall \varepsilon < \frac{1}{2}$, $B(\varepsilon, \varepsilon/2) \subseteq S$

S is bounded if $\exists\, r$ s.t. $S \subseteq B(0,r)$.

A halfspace is not bounded

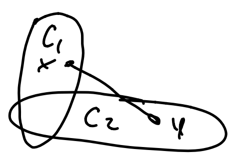
$S \subseteq \mathbb{R}^d$ is <u>compact</u> if it is closed and bounded.
Might say what it means later

Intersections: if C_1 and C_2 are convex,
so is $C_1 \cap C_2$

The empty set is convex,
as is a singleton.

A polyhedron: $x : Ax \le b$.

Unions <u>DO NOT</u> preserve convexity



$C_1 \cup C_2$ not necessarily convex

Given $x_1, .., x_k \in \mathbb{R}^d$, a <u>convex combination</u> of $x_1, ..., x_k$ is a point $x = \sum_i t_i x_i$

where $\sum_i t_i = 1$ and $t \ge 0$.

The convex hull, written $CH(x_1 ... x_k)$ is the set of all convex combinations

$$\left\{ \sum_i t_i x_i \; : \; t \ge 0, \; 1^T t = 1 \right\}$$

This is always convex.

<u>proof</u> If $y = \sum r_i x_i \quad r \geq 0 \quad 1^T r = 1$

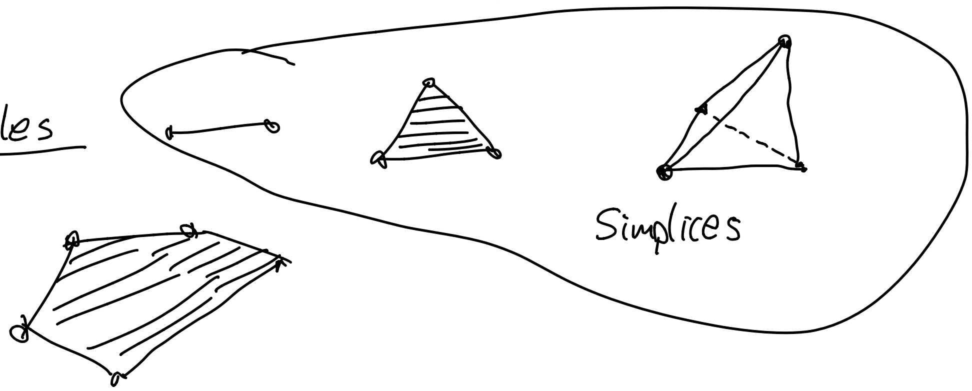
$z = \sum s_i x_i \quad s \geq 0 \quad 1^T s = 1$

and $t \in [0,1]$

$ty + (1-t)z = \sum u_i x_i$

with $u_i = t r_i + (1-t) s_i$

<u>Examples</u>



We say $x_0, \dots, x_k$ are <u>affinely independent</u>

if $x_1 - x_0, \dots, x_k - x_0$ are linearly independent.

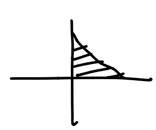
Is equivalent to $\begin{pmatrix} x_0 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} x_k \\ 1 \end{pmatrix}$ being independent.

If $x_0, \dots, x_k$ are affinely independent

then $CH(x_0, \dots, x_k)$ is a simplex.

The standard simplex in $\mathbb{R}^d$ is x s.t. $x \geq 0$

$1^T x \leq 1$



The probability simplex in $\mathbb{R}^d$ is x s.t. $x \geq 0$

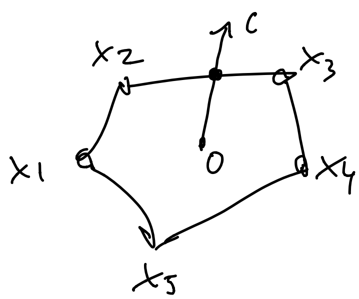
$$1^T x = 1.$$

Dan's Favorite LP:

given $x_1, \dots, x_m \in \mathbb{R}^d$ s.t. $0 \in CH(x_1, \dots, x_m)$

and $c \in \mathbb{R}^d$, $\quad \max \alpha$

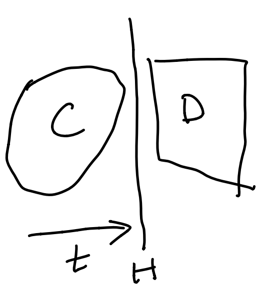
$\quad$ s.t. $\alpha c \in CH(x_1, \dots, x_m)$



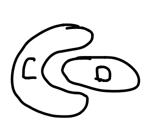
Separating Hyperplane Thm 1

If C and D are disjoint closed convex sets and C is compact, then exists a hyperplane that separates them.

That is, t s.t. $\forall x \in C$ and $y \in D$ $\; t^T x < t^T y$.



if not convex.



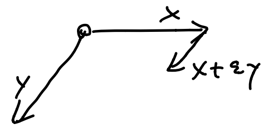
Fact If f is continuous and C is compact,

$\exists x \in C$ s.t. $f(y) \geq f(x)$ $\forall y \in C$.

"achieves its minimum"

Lem If $x^T y < 0$ then for all $0 < \varepsilon < \frac{\|y\|^2}{-x^T y}$

$\|x + \varepsilon y\|_2^2 < \|x\|_2^2$.



proof $\|x + \varepsilon y\|_2^2 = \|x\|^2 + \varepsilon x^T y + \varepsilon^2 \|y\|^2$

This is $< \|x\|_2^2$ if $\varepsilon x^T y + \varepsilon^2 \|y\|^2 < 0$

$\iff x^T y + \varepsilon \|y\|^2 < 0 \qquad (\varepsilon > 0)$

$\iff \varepsilon \leq \|y\|^2 / (-x^T y)$

proof of Thm 1

Let $c \in C$ and $d \in D$ minimize $\|c - d\|$

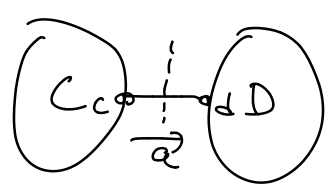
$\exists$ $c \in C$ that minimizes $\text{dist}(c, D)$ because C compact.

To prove is a d, look in $d \cap B(c, \text{dist}(c, D))$,

which is also compact.

As C and D are disjoint, $\|c - d\| \neq 0$.

Let $a = d - c$.



$a^T c = d^T c - \|c\|_2^2$

$a^T d = \|d\|_2^2 - d^T c$

Claim $\forall x \in C$, $a^T x \leq a^T c$ and $\forall x \in D$, $a^T x \geq a^T d$

$a^T c \neq a^T d$ because $a^T d - a^T c = \|d-c\|_2^2 > 0$.

<u>proof.</u> let $x \in C$. So, $\forall t \in [0,1]$ $tx + (1-t)c \in C$

$$tx + (1-t)c = c + t(x-c)$$

As $\mathrm{dist}(c + t(x-c), d) \geq \mathrm{dist}(c,d)$ for all $t \in [0,1]$,

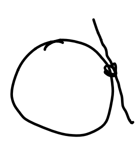
$$\|(c-d) + t(x-c)\| \geq \|c-d\|$$

$$\text{Lem 1} \Rightarrow (x-c)^T(c-d) \geq 0$$
$$\Rightarrow (x-c)^T a \leq 0$$
$$\Rightarrow x^T a \leq c^T a$$

The case of D is similar.

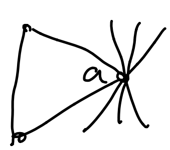
Supporting Hyperplane Theorem:

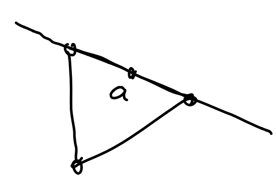
For all convex C and $a \in \text{boundary}(C)$,
$\exists\, t \neq 0$ such that $\forall x \in C$, $t^T x \leq t^T a$.



Hyperplane through a with C on one side.

$H = \{x : t^T x = t^T a\}$ is the supporting hyperplane at a





Proof idea:

For each $\varepsilon > 0$, let $a_\varepsilon \in B(a, \varepsilon)$ but $a_\varepsilon \notin C$.

Is a hyperplane that separates a_ε from C.

Take a limit of these.

If there is time, one last convex set:

Positive Semidefinite Matrices:

$A \in S_+^n$ if A is $n \times n$, symmetric, and

for all x $x^T A x \geq 0$.

If $A \in S_+^n$, can prove it by writing $A = LL^T$ – the Cholesky Factorization.

If A is symmetric, but $A \notin S_+^n$,

let $A = V \Lambda V^T$ be spectral decomposition and $\lambda_n < 0$.

A hyperplane separating A from S_+^n is given by

$\{$ symmetric X : $v_n^T X v_n = 0 \}$

$v_n^T X v_n \geq 0$ for $X \in S_+^n$. $v_n^T A v_n = \lambda_n < 0$,

And is a hyperplane because

$$v^T X v = \sum_{1 \leq i, j \leq n} X(i,j)\, v(i)\, v(j) \quad \text{is linear in } X.$$