Linear Programming & Convex Sets.

An LP has variables $x \in \mathbb{R}^d$, an objective function $\max c^T x$, specified by $c \in \mathbb{R}^d$, subject to $m$ constraints $a_i^T x \leq b_i$, where $a_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$, for $1 \leq i \leq m$.

We typically collect the $a_i$ into an $m$-by-$d$ matrix $A$, and the $b_i$ into an $m$-vector $b$, and write

$$Ax \leq b,$$

where this means the inequality holds in every coordinate:

$$x \leq y \iff \text{for all } i \ x(i) \leq y(i).$$

We will see many forms of linear programs.

If want $a^T x = b$, write it as $(-a)^T x \leq b$.

If want $a^T x \leq b$, write it as $a^T x \leq b$ and $(-a)^T x \leq -b$

The set $\{x : Ax \leq b\}$ is a polyhedron.

In particular, it is convex.

We will spend most of this lecture understanding convex sets.
Def: \( C \subseteq \mathbb{R}^n \) is convex if for all \( x, y \in C \) and all \( 0 \leq t \leq 1 \), \( tx + (1-t)y \in C \).

That is, the line segment from \( x \) to \( y \) is in \( C \).

**Examples**

A subspace: \( \{ x : Ax = 0 \} \)

\[ A\!x = 0 \text{ and } A\!y = 0 \implies A( tx + (1-t)y ) = 0. \]

An affine space: \( \{ x : Ax = b \} \)

\[ A\!x = b \text{ and } A\!y = b \implies A( tx + (1-t)y ) = tA\!x + (1-t)A\!y = tb + (1-t)b = b. \]

A line segment: \( \{ x : a \leq x \leq b \} \), \( a, b \in \mathbb{R} \)

An open line segment: \( \{ x : a < x < b \} \), \( a, b \in \mathbb{R} \)

Positive reals: \( x > 0 \)

A halfspace: \( x \in \mathbb{R}^d \) s.t. \( a^\top x \leq b \)

\[ a^\top( tx + (1-t)y ) = ta^\top x + (1-t)a^\top y \]

\[ \leq tb + (1-t)b = b \]

An open halfspace: \( x \in \mathbb{R}^d \) s.t. \( a^\top x < b \)
A norm ball: $\{x : \|x\| \leq 1\}$, for any norm $\|\cdot\|$

Let $B(x, r)$ be the ball of radius $r$ around $x$

$$= \{y : \|x - y\| \leq r\}$$

Some topology

For $S \subseteq \mathbb{R}^d$, $x$ is in the interior of $S$ if $\exists \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq S$

Ex:

$x$ is on the boundary of $S$ if $\forall \varepsilon > 0$, $B(x, \varepsilon)$ contains points in $S$ and not in $S$

If $S = \{x : 0 < x < 1\}$, the boundary = $\{0, 1\}$

$S$ is closed if boundary $(S) \subseteq S$

$S$ is open if $S = \text{interior}(S)$

e.g. $\{x : 0 < x < 1\}$ with $\varepsilon \leq \frac{1}{2}$, $B(x, \varepsilon) \subseteq S$

$S$ is bounded if $\exists r$ s.t. $S \subseteq B(0, r)$.

A halfspace is not bounded.
$S \subseteq \mathbb{R}^d$ is **compact** if it is closed and bounded.

Might say what it means later.

Intersections: if $C_1$ and $C_2$ are convex, so is $C_1 \cap C_2$.

The empty set is convex, as is a singleton.

A polyhedron: $x \colon Ax \leq b$.

Unions **DO NOT** preserve convexity.

Given $x_1, \ldots, x_k \in \mathbb{R}^d$, a **convex combination** of $x_1, \ldots, x_k$ is a point $x = \sum_i t_i x_i$ where $\sum_i t_i = 1$ and $t_i \geq 0$.

The convex hull, written $CH(x_1, \ldots, x_k)$ is the set of all convex combinations

\[
\{ \sum_i t_i x_i : t_i \geq 0, \sum_i t_i = 1 \}
\]

This is always convex.
proof If \( y = \sum \gamma_i x_i \) \( \gamma \geq 0 \) \( 1^\top \gamma = 1 \)
\[
z = \sum s_i x_i \quad s \geq 0 \quad 1^\top s = 1
\]
and \( t \in [0,1] \)
\[
t y + (1-t) z = \sum u_i x_i
\]
with \( u_i = t \gamma_i + (1-t) s_i \).

Examples

Simplices

We say \( x_0, \ldots, x_k \) are **affinely independent**
if \( x_i - x_0, \ldots, x_k - x_0 \) are linearly independent.
Is equivalent to \( (x_0)_{1 \leq i \leq k} \) being independent.

If \( x_0, \ldots, x_k \) are affinely independent
then \( \mathbf{C}^k(x_0, \ldots, x_k) \) is a simplex.

The standard simplex in \( \mathbb{R}^d \) is \( x \) s.t. \( x \geq 0 \)
\[
1^\top x = 1
\]
The probability simplex in $\mathbb{R}^d$ is $x$ s.t. $x \geq 0$ and $1^T x = 1$.

Dan's Favorite LP:

given $x_1, \ldots, x_m \in \mathbb{R}^d$ s.t. $0 \in CH(x_1, \ldots, x_m)$
and $c \in \mathbb{R}^d$, $\max c$ s.t. $d \in CH(x_i, \ldots, x_m)$

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**Separating Hyperplane Thm 1**

If $C$ and $D$ are disjoint closed convex sets and $C$ is compact, then exists a hyperplane that separates them.

That is, $t$ s.t. $x \in C$ and $y \in D$ if not convex.

If not convex.

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$C$ | $D$
Fact: If \( f \) is continuous and \( C \) is compact, 
\[ \exists x \in C \text{ s.t. } f(y) = f(x) \quad \forall y \in C. \]

"achieves its minimum"

Lemma: If \( x^T y < 0 \) then for all \( 0 < \varepsilon < \frac{\|y\|^2}{-x^T y} \)
\[ \|x + \varepsilon y\|_2^2 < \|x\|_2^2. \]

\[ \begin{align*}
\text{proof:} \quad \|x + \varepsilon y\|_2^2 &= \|x\|^2 + 2x^T y + \varepsilon^2 \|y\|^2 \\
&< \|x\|^2 \quad \text{if} \quad x^T y + \varepsilon \|y\|^2 < 0 \\
&\iff \quad x^T y + \varepsilon \|y\|^2 < 0 \quad (\varepsilon > 0) \\
&\iff \quad \varepsilon < \frac{\|y\|^2}{-x^T y}
\end{align*} \]

Proof of Thm 1:

Let \( c \in C \) and \( d \in D \) minimize \( \|c - d\| \)

\( \exists \ c \in C \) that minimizes \( \text{dist}(c, d) \) because \( C \) compact.

To prove is \( a \), look in \( d \cap B(C, \text{dist}(c, d)) \),

which is also compact.

As \( C \) and \( D \) are disjoint, \( \|c - d\| \neq 0. \)

Let \( a = d - c. \)
\[ \begin{align*}
a^T c &= d^T c - \|c\|_2^2 \\
a^T d &= \|d\|_2^2 - d^T c
\end{align*} \]

Claim \( \forall x \in C, a^T x = a^T c \) and \( \forall x \in D, a^T x = a^T d \)
\( \alpha^T c + \alpha^T d \) because \( \alpha^T d - \alpha^T c = \|d - c\|_2^2 > 0. \)

**Proof:** Let \( x \in C \). So, \( \alpha + (1-t)c \in C \)
\[ c + t(x-c) \]
As \( \text{dist}(C + t(x-c), d) = \text{dist}(C, d) \) for all \( t \in [0, 1] \),
\[ \|c - d\| + t(x-c) \| = \|c - d\| \]

**Lemma:**

\[ \Rightarrow (x - c)^T (c - d) \geq 0 \]
\[ \Rightarrow (x - c)^T \alpha \leq 0 \]
\[ \Rightarrow x^T \alpha \leq c^T \alpha \]

The case of \( D \) is similar.

**Supporting Hyperplane Theorem:**

For all convex \( C \) and \( a \in \text{boundary}(C) \),
exists \( t \neq 0 \) such that \( \forall x \in C, \ t^T x \leq t^T a \).

Hyperplane through \( a \) with \( C \) on one side.

\( H = \{ x : t^T x = t^T a \} \) is the supporting hyperplane at \( a \).
Proof idea:
For each $\varepsilon > 0$, let $a_\varepsilon \in B(q, \varepsilon)$ but $a_\varepsilon \notin C$. Is a hyperplane that separates $a_\varepsilon$ from $C$.
Take a limit of these.

If there is time, one last convex set:
Positive Semidefinite Matrices:
$A \in S^+_n$ if $A$ is non-symmetric and
for all $x \in \mathbb{R}^n A x = 0$.

If $A \in S^+_n$, can prove it by writing $A = LL^T$ — the Cholesky Factorization.

If $A$ is symmetric, but $A \notin S^+_n$,
let $A = V \Lambda V^T$ be spectral decomposition and $\lambda_n < 0$.
A hyperplane separating $A$ from $S^+_n$ is given by
\[ \{ \text{symmetric } X : \mu^T X \nu = 0 \} \]
\[ \mu^T X \nu \geq 0 \text{ for } X \in S^+_n. \quad \mu^T A \mu = \lambda_n < 0, \]
And is a hyperplane because
\[ \mu^T X \nu = \sum_{i \leq j} X(i, j) \mu(i) \nu(j) \] is linear in $X$. 