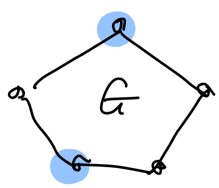
Max Cut: We will use convex programming (linear programming over semidefinite matrices) to get a 0.878... approximation of maxcut. This is a famous result of Goemans & Williamson

Input: graph $G=(V,E)$.

For $S \subset V$, define $cut(S) = \#\{(a,b) \text{ s.t. } |\{a,b\} \cap S| = 1\}$

$$maxcut(G) = \max_S cut(S).$$

Ex.



$maxcut(G) = 4$.

In our previous notation, $cut(S) = |\partial(S)|$

Easy results first. Define $m = |E|$.

Lem 1 $maxcut(G) \geq m/2$

proof Consider choosing S uniformly at random.

$\Pr[a \in S] = \frac{1}{2}$ for every $a \in V$.

For each edge (a,b)

$$\Pr[(a,b) \in \partial(S)] = \Pr[a \in S \text{ and } b \notin S] + \Pr[a \notin S \text{ and } b \in S]$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So, $\mathop{\mathbb{E}}_S cut(S) = \sum_{(a,b)\in E} \Pr[(a,b) \in \partial(S)] = \frac{m}{2}$

As max $\geq$ average, $\exists\, S$ s.t. $cut(S) \geq m/2$.

Can turn this into an algorithm. But there's a simpler algorithm.

Let's describe it in terms of $\{\pm 1\}^n$ vectors, where $n = |V|$.

Local Search

Start with any $x \in \{\pm 1\}^n$ (like $\mathbb{1}$, or random)

while $\exists a$ s.t. $x(a) \sum_{b:(a,b)\in E} x(b) > 0$ (most neighbors on same side)

$x(a) = -x(a)$.

Return x (or $S_x = \{a : x(a) = 1\}$)

Idea: if moving a into or out of S increases the cut, do it.

Claim 1 $\quad \mathrm{cut}(S_x) = \frac{1}{2} \sum_{(a,b)\in E} 1 - x(a)x(b)$

proof $\quad x(a)x(b) = -1$ if $(a,b) \in \partial(S_x)$
$\qquad\qquad = 1$ o.w.

Claim 2 $\quad$ If $\hat{x}$ is vector after moving a,

$$\mathrm{cut}(S_{\hat{x}}) = \mathrm{cut}(S_x) + x(a) \sum_{b:(a,b)\in E} x(b)$$

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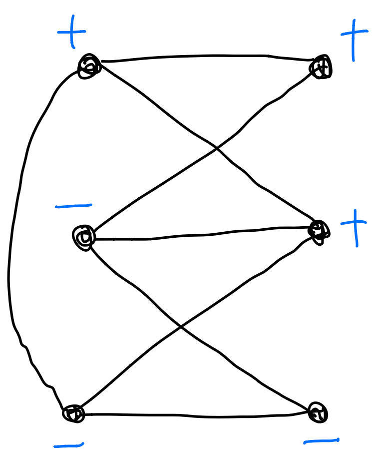

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$$(1) - (1) \quad \longrightarrow \quad (1) \rightarrow (-1)$$

$cut(S_x) = 0$ $\qquad\qquad$ $cut(S_x) = 1$

<u>Lem 2</u> Local Search terminates, and returns x with $cut(S_x) \geq \frac{m}{2}$

<u>proof</u> Claim 2 implies $cut(S_x)$ increases by at least 1 at every step.

When algorithm stops, $\forall a \; \sum_{b:(a,b)\in E} x(a)x(b) \leq 0$.

$$cut(S_x) = \frac{1}{2}\sum_{(a,b)\in E} 1 - x(a)\,x(b) = \frac{m}{2} - \frac{1}{2}\sum_{(a,b)\in E} x(a)\,x(b)$$

and, $$\sum_{(a,b)\in E} x(a)x(b) = \frac{1}{2}\sum_{a\in U}\ \sum_{b:(a,b)\in E} x(a)x(b) \leq 0$$

How to do better?

Goemans & Williamson '95: relax $x(a) \in \{\pm 1\}$

replace with $v_a \in \mathbb{R}^n$, $\|v_a\|_2 = 1$

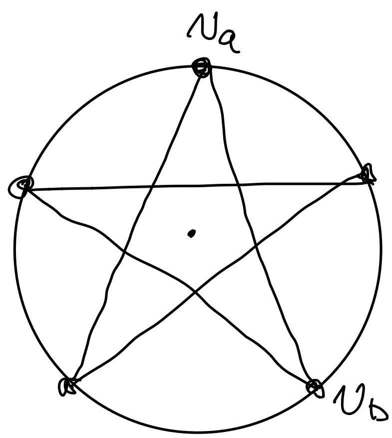
$$x(a)x(b) \rightarrow v_a^T v_b$$

$$\sum_{(a,b)\in E} 1 - x(a)x(b) \rightarrow \sum_{(a,b)\in E} 1 - v_a^T v_b$$

Solve the vector problem

$$VP(G) = \max \sum_{(a,b)\in E} 1 - v_a^T v_b \quad \text{s.t. } \|v_a\|_2 = 1$$

Ex.



$v_a^T v_b = \cos\left(\frac{4\pi}{5}\right) \approx -0.81$

for all $(a,b) \in E$

$$\frac{1}{2} \sum_{(a,b)\in E} 1 - v_a^T v_b \approx 4.52 > \text{maxcut}(G) = 4$$

1. We can turn the solution into an approximate solution to maxcut.
2. We can approximately solve VP in polynomial time.

Claim 3 $\mathrm{VP}(G) \geq \mathrm{maxcut}(G)$

proof consider $v_a = u \cdot x(a)$ for any unit vector u.

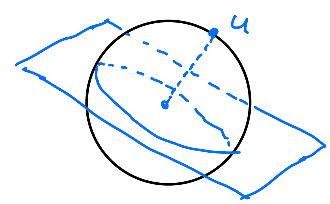
Now, $v_a^T v_b = x(a) x(b)$.

But we can choose v_a differently, and get a larger value.

To round vectors $v_a, \dots, v_n$ into ± 1 $x(1), \dots, x(n)$:

Choose a random unit vector u,

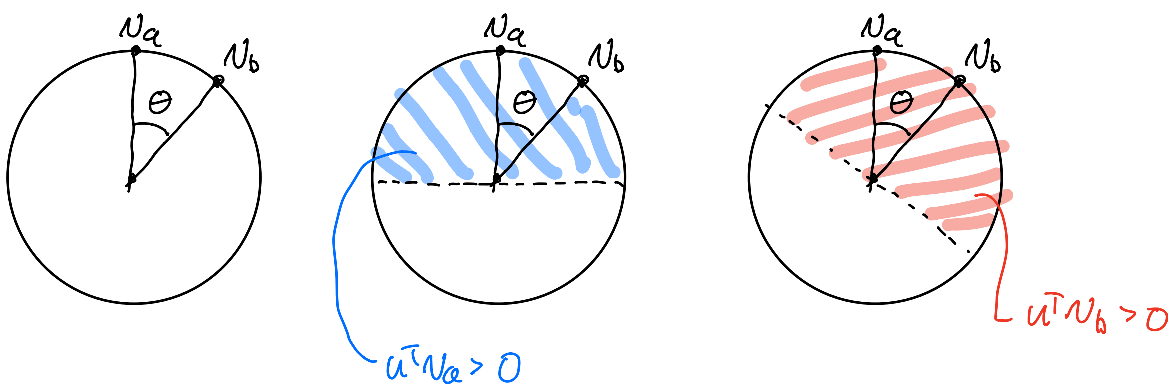
set $x(a) = \begin{cases} 1 & \text{if } u^T v_a \geq 0 \\ -1 & \text{o.w.} \end{cases}$

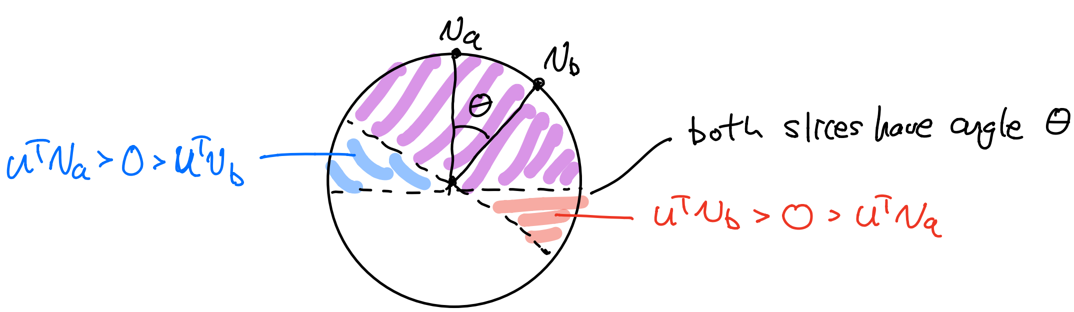


(is equivalent to use Gaussian random u)

Claim 4 $\Pr[(a,b) \in \partial(S_x)] = \frac{1}{\pi} \operatorname{ang}(v_a, v_b)$

proof First, look at the 2D case





In general, project u to $\text{span}(v_a, v_b)$, and apply this analysis.

<u>Claim 5</u> $\mathbb{E}_u \, \text{cut}(S_x) = \frac{1}{\pi} \sum_{(a,b) \in E} \text{acos}(v_a^T v_b)$

<u>proof</u> $v_a^T v_b = \cos(\theta)$, so $\theta = \text{acos}(v_a^T v_b)$

<u>Claim 6</u> $\min_{-1 \le t \le 1} \dfrac{\frac{1}{\pi} \text{acos}(t)}{\frac{1}{2}(1-t)} \ge 0.878$

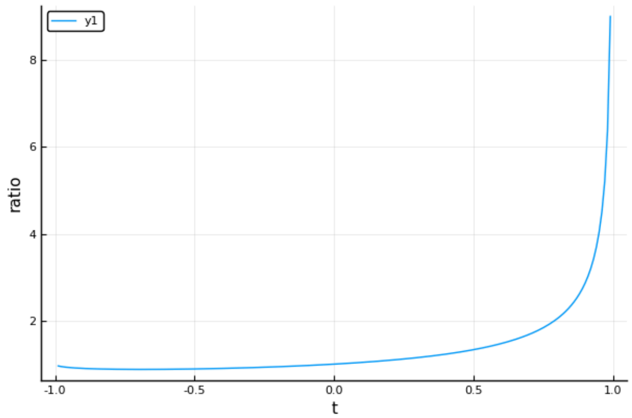
<u>Theorem</u> $\mathbb{E}_u \, \text{cut}(S_x) \ge 0.878 \cdot \text{maxcut}(G)$

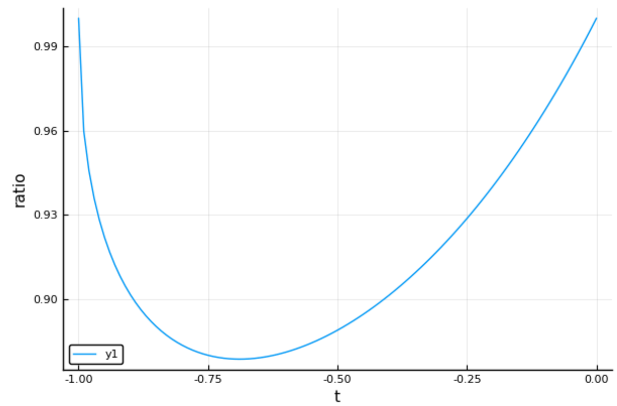
<u>proof</u>
$$
\begin{aligned}
\mathbb{E}_u \, \text{cut}(S_x) &\ge \frac{1}{\pi} \sum_{(a,b) \in E} \text{acos}(v_a^T v_b) \\
&\ge 0.878 \cdot \frac{1}{2} \sum_{(a,b) \in E} 1 - v_a^T v_b \\
&= 0.878 \cdot VP(G) \\
&\ge 0.878 \cdot \text{maxcut}(G)
\end{aligned}
$$

<u>Proof of claim 6</u>

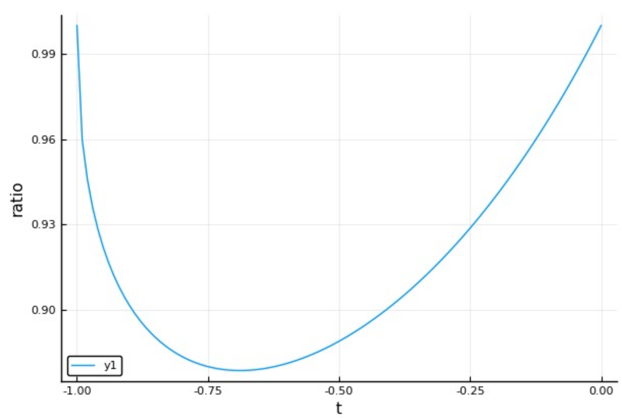
1. Change variables to $t = \cos(\theta)$.
   Set derivative in θ to 0.
   Find minimum where $\cos(\theta) + \theta \sin(\theta) = 1$.

or, 2. plot it.



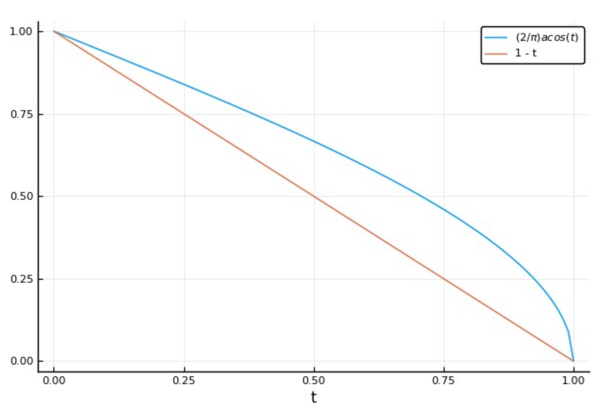


3. Use the plot to derive the bound. (see GW)



here, ratio is convex.

So use a supporting plane to get a lower bound



here, ratio ≥ 1

How to solve VP:

$$\max_{v_1,\dots,v_n} \frac{1}{2}\sum_{(a,b)} 1 - v_a^T v_b \quad \text{s.t. } v_a^T v_a = 1, \text{ for all } a.$$

This problem is linear in the Gram matrix:

$M(a,b) = v_a^T v_b \qquad M = V^T V$ where $V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

problem becomes $\max \frac{1}{2}\sum_{(a,b)\in E} 1 - M(a,b)$ s.t. $M(a,a) = 1,\ \forall a$

and M is a Gram matrix.

Claim:

M is a Gram matrix iff M is positive semidefinite.

proof $M = V^T V \Rightarrow x^T M x = x^T V^T V x \geq 0,\ \forall x$

and if M is psd, can find a Cholesky Factorization,

$M = LL^T$,

so, M is the Gram matrix of $V = L^T$.

So, solve

$$\max \frac{1}{2}\sum_{(a,b)\in E} 1 - M(a,b) \quad \text{s.t. } M \succeq 0,\ M(a,a) = 1,\ \forall a$$

Cholesky factor $M = LL^T$

$x = \text{sign}(u^T V)$ for a random vector u