Sparse recovery 2: Recovering sparse signals with noise.

Problem: Given d-by-n A, $d \ll n$ $\left(\quad A \quad \right)$
and a vector $b \in \mathbb{R}^d$, try to find a
sparse x s.t. $Ax \approx b$.

Formally:

Given $\varepsilon > 0$, $\min \|x\|_0$ s.t. $\|Ax - b\|_2 \le \varepsilon$ (P_0)

Is NP-hard, but can approximately solve it for nice A.

Let x_* be the solution to
$\min \|x\|_1$ s.t. $\|Ax - b\|_2 \le \varepsilon$. (P_1)
We will show x_* is close to the solution to P_0.

think $d = \text{const} \cdot n$
$s = \text{const} \cdot d$

slight variant: usually stated with squares

Nice: A satisfies the (s, δ) - Restricted Isometry Property if for all x s.t. $\|x\|_0 \le s$

$$(1-\delta)\|x\|_2 \le \|Ax\|_2 \le (1+\delta)\|x\|_2$$ think $\delta \ll 1/4$

<u>Def</u> $\delta_s(A)$ = smallest δ s.t. A satisfies (s, δ)-RIP

Theorem If $\delta_{3s} + \frac{1}{\sqrt{2}}\delta_{2s} \le 1 - 1/\sqrt{2}$, $b = Ax_0 + e$
with $\|x_0\|_0 \le s$ and $\|e\|_2 \le \varepsilon$,
then x_*, the solution to P_1, satisfies

$$\|x_0 - x_*\|_2 \le \frac{2\sqrt{2}\,\varepsilon}{1 - 1/\sqrt{2} - \delta_{3s} - \frac{1}{\sqrt{2}}\delta_{2s}}$$

Notation: let $S = \mathrm{supp}(x_0)$, $s = |S|$, $T = \{1,\dots,n\} - S$.
$h = x_* - x_0$ Goal: show $\|h\|_2 \le ?$

Claim 1 $\|Ah\|_2 \le 2\varepsilon$ use RIP to get $\|h\|_2 \le ?\,\|Ah\|_2$

proof $Ah = Ax_* - Ax_0 = (Ax_* - b) + (Ax_0 - b)$
so $\|Ah\|_2 \le \|Ax_* - b\|_2 + \|Ax_0 - b\|_2 \le 2\varepsilon$

Let $h(S)$ be the restriction of h to coordinates in S.
(or just zero out coordinates not in S)

Claim 2 $\|h(S)\|_1 \ge \|h(T)\|_1$

proof

$\|x_0\|_1 \ge \|x_*\|_1$, because x_* solves $\min \|x\|_1$ s.t. ....

$$
\begin{aligned}
&= \|x_0 + h\|_1 = \|(x_0+h)(S)\|_1 + \|(x_0+h)(T)\|_1 \\
&\qquad\qquad \swarrow \Delta\text{-ineq} \qquad\qquad \downarrow =,\ x_0(T) = 0 \\
&\ge \|x_0(S)\|_1 - \|h(S)\|_1 + \|h(T)\|_1 \\
&= \|x_0\|_1 - \|h(S)\|_1 + \|h(T)\|_1
\end{aligned}
$$

$\Rightarrow \|h(S)\|_1 \ge \|h(T)\|_1$

Would also like to say $\|h(S)\|_2 \geq \text{const} \cdot \|h(T)\|_2$,

but might not be true.

Ex. $h(S) = \frac{1}{s}, \ldots, \frac{1}{s}$, so $\|h(S)\|_1 = 1$, $\|h(S)\|_2 = 1/\sqrt{s}$

$h(T) = 1, 0, \ldots, 0$, so $\|h(T)\|_1 = 1$, $\|h(T)\|_2 = 1$

Will show something like this after excluding $3s$ coordinates from T.

Def. T_1 = the $2s$ coords in T on which h is largest

$R = T - T_1 = \{1, \ldots, n\} - S - T_1$

Claim 3 $\|h(R)\|_2^2 \leq \frac{1}{2} \|h(S)\|_2^2$ (only uses claim 2)

proof Recall for every vector v, $\|v\|_2^2 \leq \|v\|_1 \|v\|_\infty$

$$\left(\sum_i v(i)^2 \leq \sum_i \|v\|_\infty |v(i)| = \|v\|_\infty \|v\|_1 \right)$$

$$\|h(R)\|_\infty \leq \min_{i \in T_1} |h(i)| \leq \frac{1}{2s} \|h(T_1)\|_1 \quad (\min \leq \text{avg})$$

$$\|h(R)\|_1 \leq \|h(T)\|_1 \quad (\text{as } R \subseteq T)$$

$$\Rightarrow \|h(R)\|_2^2 \leq \frac{1}{2s} \|h(T)\|_1^2 \leq \frac{1}{2s} \|h(S)\|_1^2$$

$$\leq \frac{s}{2s} \|h(S)\|_2^2,$$

as $\|h(S)\|_0 = s$

We also want to show $\|A\, h(R)\|_2 \leq$ something.

Will need RIP, but only applies to sparse vectors.

Let T_2 = largest $2s$ coords of h in $T - T_1$

T_3 = " " " " " " $T - T_1 - T_2$

etc.

So, $T = T_1 \cup T_2 \cup \cdots \cup T_k$, some k, and T_k can be smaller.

$R = T_2 \cup \cdots \cup T_k$

<u>Claim 4</u> $\sum_{i \geq 2} \| h(T_i) \|_2 \leq 1/\sqrt{2} \, \| h(S) \|_2$ (wts $\|h\|_2 \leq$?)

<u>proof</u> For $i \geq 2$,

$$\| h(T_i) \|_2^2 \leq \| h(T_i) \|_\infty \, \| h(T_i) \|_1$$

$$\leq \| h(T_i) \|_\infty \, \| h(T_{i-1}) \|_1, \quad \text{as } h \text{ decreases with } i$$

$$\leq \frac{1}{2s} \| h(T_{i-1}) \|_1^2, \quad \text{as min} \leq \text{average}$$

$$\Rightarrow \sum_{i=2}^{k} \| h(T_i) \|_2 \leq (2s)^{-1/2} \sum_{i=1}^{k-1} \| h(T_{i-1}) \|_1$$

$$= (2s)^{-1/2} \| h(T_1 \cup \cdots \cup T_{k-1}) \|_1$$

$$\leq (2s)^{-1/2} \| h(T) \|_1$$

$$\leq (2s)^{-1/2} \| h(S) \|_1 \quad \text{(by claim 2)}$$

$$\leq 1/\sqrt{2} \, \| h(S) \|_2, \quad \text{as } \| h(S) \|_0 = s$$

<u>Claim 5</u> $\| A \, h(R) \|_2 \leq \frac{1}{\sqrt{2}} (1 + \delta_{2s}) \, \| h(S) \|_2$

$$\| A \, h(R) \|_2 \underset{\triangle \text{ ineq}}{\leq} \sum_{i \geq 2} \| A \, h(T_i) \|_2 \underset{\text{RIP, } |T_i| \leq 2s}{\leq} (1 + \delta_{2s}) \sum_{i \geq 2} \| h(T_i) \|_2 \quad \text{(claim 4)}$$

$$\leq \frac{1}{\sqrt{2}} (1 + \delta_{2s}) \, \| h(S) \|_2$$

Proof of theorem

$\{1,\ldots,n\} = S \cup T_1 \cup R$

$$\|h\|_2 = \|x_0 - x_*\|_2 \le \frac{2\sqrt{2}\,\varepsilon}{1 - 1/\sqrt{2} - \delta_{2s} - \frac{1}{\sqrt{2}}\delta_{2s}}$$

We know $\|Ah\|_2 \le 2\varepsilon$

and $\|A\,h(R)\|_2 \le \frac{1}{\sqrt{2}}(1+\delta_{2s})\,\|h(S)\|_2 \le \frac{1}{\sqrt{2}}(1+\delta_{2s})\|h(S\cup T_1)\|_2$

Δ-inequality: $2\varepsilon \ge \|Ah\|_2 \ge \|A\,h(S\cup T_1)\|_2 - \|A\,h(R)\|_2$

$$\|A\,h(S\cup T_1)\|_2 \ge (1-\delta_{3s})\,\|h(S\cup T_1)\|_2 \quad \text{, by RIP}$$

$$\Rightarrow \|Ah\|_2 \ge \left[(1-\delta_{3s}) - \tfrac{1}{\sqrt{2}}(1+\delta_{2s})\right]\|h(S\cup T_1)\|_2$$

Claim $\|h(S\cup T_1)\|_2 \ge \frac{1}{\sqrt{2}}\|h\|_2$, because

$$\|h(S\cup T_1)\|_2^2 = \|h\|_2^2 - \|h(R)\|_2^2 \ge \|h\|_2^2 - \tfrac{1}{2}\|h(S)\|_2^2 \ge \tfrac{1}{2}\|h\|_2^2$$

So, $\|Ah\|_2 \ge \frac{1}{\sqrt{2}}\left[(1-\delta_{3s}) - \frac{1}{\sqrt{2}}(1+\delta_{2s})\right]\|h\|_2$

$$\Rightarrow \|h\|_2 \le \frac{2\sqrt{2}\,\varepsilon}{1 - 1/\sqrt{2} - \delta_{2s} - \frac{1}{\sqrt{2}}\delta_{2s}}$$