Sparse Solutions to $Ax = b$

Are NP-hard to find

But we want them

And for special $A$ we can find them

Today: exact solutions

Thursday: approximate solutions.

**Def.** For $x \in \mathbb{R}^n$, $\text{supp}(x) = \{ i \mid x(i) \neq 0 \}$

$$||x||_0 = |\text{supp}(x)|$$

Recall, $|| \cdot ||_0$ satisfies $||x+y||_0 \leq ||x||_0 + ||y||_0$,

but it is not a norm because $||cx||_0 = ||x||_0$ for $c \neq 0$.

**Lem.** If $||x||_0 = S$, then $||x||_1 \leq S ||x||_2 \leq S ||x||_0$

Sparse solutions to $Ax = b$:

$$\min_x ||x||_0 \text{ s.t. } Ax = b \quad (*)$$

find $x$ s.t. $Ax = b$ and $||x||_0 \leq S$

$$\min_x ||Ax - b||_2 \text{ s.t. } ||x||_0 \leq S$$

$$\min_x ||Ax - b||_2 + \lambda ||x||_0$$
Why?

To regularize \((Ax) = b\)

when \(\dim x \gg \dim b\).

Sparse Regression:
want a solution with \(\|x\|_0\) small because
isolates dominant effects
easier to interpret

Signal processing: sounds and images are
often sparse + noise
in Fourier or wavelet basis.

Compressed sensing:
Assuming signal is sparse,
design measurement matrix \(A\)
that allows for approximate recovery
from few measurements
We will consider conditions on $A$ and $b$ that make these problems solvable.

The **Kruskal Rank** of $A$ is the largest number $r$ such that every set of $r$ columns are independent.

\[ \iff \forall \neq 0, \ |x|_1 \leq r \implies Ax \neq 0 \]

**Claim** If $A$ has Kruskal Rank $r$, $Ax = b$, and $|x|_1 \leq \sqrt{r}$, then $x$ is the unique solution to $(\ast)$.

**Proof** Assume, by way of contradiction, there is a $y \neq x$ such that $Ay = b$ and $|y|_1 \leq \sqrt{r}$.

Then $A(x - y) = 0$, but $|x - y|_1 \leq |x|_1 + |y|_1 = 2r$, and $x - y \neq 0$, a contradiction.

But, it is NP-hard to compute the Kruskal rank. That is, it is hard to check if there is a $x$ such that $Ax = 0$ and $|x|_1 \leq k$.

**Incoherence** Let the columns of $A$ be $a_1, \ldots, a_n$. $A$ is $\mu$-incoherent if for all $i \neq j$,

\[ |a_i^T a_j| \leq \mu \|a_i\|_2 \|a_j\|_2 \]

We can compute the incoherence!
Claim: If $A$ is $d$-by-$n$ Gaussian random matrix, is probably $\mu$-incoherent with $\mu \leq O(\sqrt{(d \log n)/d})$

Sketch: $||a_{il}||_{2}$ is probably $\sim \sqrt{d}$. (each coord $\sim O(1)$)

If fix $a_{i}$, then $\frac{a_{i}^\top a_{j}}{||a_{i}||}$ is $N(0,1)$

So, probably constant and chance $> \sqrt{2\ln n} \leq \frac{1}{n^2}$

Henceforth, assume $||a_{i}||_{2} = 1$ for all $i$.

So $\mu$-incoherent $\Rightarrow ||a_{i}^\top a_{j}|| \leq \mu$, $\forall i \neq j$

Lem 2 Kruskal Rank $> \frac{1}{\mu}$

Proof. Let $A$ be $\mu$-incoherent and let $Ax = 0$.

Let $S = \text{supp}(x)$, $s = |S|$. We will show $s > \frac{1}{\mu}$.

Let $|x(i)|$ be largest absolute value, and assume wolog $x(i) = 0$.

We have $0 = a_{i}^\top Ax = a_{j}^\top a_{i} x(j) + \sum_{i \in S - \{i\}} a_{j}^\top a_{i} x(i)$

$\Rightarrow x(j) = \left| \sum_{i \in S - \{i\}} a_{j}^\top a_{i} x(i) \right| \leq \sum_{i \in S - \{i\}} |x(i)|$

As $x(i) \geq |x(i)|$, this implies $(s-1)\mu \geq 1$

$\Rightarrow s > \frac{1}{\mu}$
Orthogonal Matching Pursuit (OMP)
Input: \(A, b, s\). Output: \(x\) s.t. \(\|x\|_0 \leq S\).
Init: \(x_0 = 0\), \(S = \emptyset\), \(r = b\)
While \(\|r\|_2 > 0\)
\[ j = \arg \max_j |a_j^Tr| \]
\[ S = S \cup \{j\} \]
\[ T = \text{projection of } b \text{ orthogonal to } \text{Span}(a_i : i \in S) \]
Let \(x\) be the solution to \(Ax = b\), \(x(i) = 0\) for \(i \notin S\).

Thm 1. If \(A\) is \(\mu\)-incoherent and there is an \(x\) s.t. \(Ax = b\) and \(\|x\|_0 < \frac{1}{2\mu}\), then OMP returns \(x\).

Let \(T = \text{supp}(x) = \{i : x(i) \neq 0\}\).
We will show
i. each index \(j\) chosen is in \(T\), and
ii. no index is chosen twice.
These imply the theorem.

ii is easy:
\[ as \ r \perp a_i \text{ for } i \notin S, i \in S \Rightarrow a_i^Tr = 0 \]
To prove i, we show

**Lemma 3** If at the start of an iteration

\[ \gamma \in \text{span}(\{a_i : i \in T\}), \]

then algorithm picks some \( j \in T \)

First note that at start \( \gamma \in \text{span}(\{a_i : i \in T\}) \),

because \( Ax = 0 \), \( \text{supp}(x) = T \), and \( \gamma \) starts at \( 0 \).

**Proof** Let \( \gamma = \sum_{i \in T} a_i \gamma_i \). Recall \( |T| < \frac{1}{\mu} \).

Kruskal's rank \( > \frac{1}{\mu} \) and Lemma 2 imply this representation is unique.

Assume, wlog, \( y(k) \) has the largest absolute value.

To show that the algorithm picks a \( j \in T \),

it suffices to show there is a \( j \in T \) s.t.

\[ |a_j^\top \gamma| > |a_h^\top \gamma| \]

for all \( h \notin T \).

We will show this for \( j = k \).

1. For \( h \notin T \),

\[ |a_h^\top \gamma| \leq \sum_{i \in T} |a_h^\top a_i| |y(i)| \]

\[ \leq \sum_{i \in T} \mu |y(i)| = \mu |T| |y(k)| < \frac{1}{2} |y(k)| \]
2. \[ |q_k^r| = a_k^T \alpha_k |y(k)| - \sum_{i \in T \setminus \{k\}} a_k^T \alpha_i |y(i)| \]

\[ \geq |y(k)| - (|T|-1) \mu |y(k)| > \frac{1}{2} |y(k)| \]

So, \[ |q_k^r| > |q_h^r| \] for \( h \notin T \)

**Proof of Theorem 1**

At start, \( \tau = b = \sum_{i \in T} x(i) \alpha_i \).

So, Lem 3 \( \Rightarrow \) alg picks \( j \in T \)

Assuming alg has constructed \( S \subset T \),

\( \tau = \) projection of \( b \) orthogonal to \( \{ \alpha_i : i \in S \} \)

We claim \( \tau \in \text{span}(\alpha_i : i \in T) \)

**Proof 1** If \( A_S = A(:,S) \), then the projection is

\[ \tau = b - A_S (A_S^T A_S)^{-1} A_S^T \cdot b \]

**Proof 2** \( \tau = \arg \min_{\gamma} \| b - \sum_{i \in S} y(i) \alpha_i \| \)

Now, proceed by induction.

There is a fast (ish) implementation, because can maintain the projection using rank-1 updates.
For strange historical reasons, algorithms for the problems often have "pursuit" in their name.

For exact recovery, we will next examine Basis Pursuit,

$$\min_x \|x\|_1 \text{ s.t. } Ax = b \quad (**)$$

For approximate recovery, BPDN (Basis Pursuit De-Noising)

$$\min_x \|x\|_1 \text{ s.t. } \|Ax - b\|_2 \leq \varepsilon$$

This is like LASSO for sparse regression

$$\min_x \|Ax - b\|_2 \text{ s.t. } \|x\|_1 \leq t$$

Some intuition for (**)  
Solution is at least $t$ s.t. $\{x: \|x\|_1 = t\}$ hits $Ax = b$,  
Almost always happens at a low dimensional face of the ball $B_1(0,t) = \{x: \|x\|_1 = t\}$
We analyze these assuming $A$ satisfies the $(s, \delta)$-Restricted Isometry Property (RIP): 

$\exists Y$ s.t. $\|x\|_0 \leq s$

$|(1-\delta)\|x\|^2_2 \leq \|Ax\|^2_2 \leq (1+\delta)\|x\|^2_2$

$Q$ is an isometry if $\|Qx\|^2_2 = \|x\|^2_2$ for all $x$. That is, if columns are orthonormal.

The condition $|(1-\delta)\|x\|^2_2 \leq \|Ax\|^2_2 \leq (1+\delta)\|x\|^2_2$

is an approximation of this.

But we can only hope for this when $\|x\|_0 \leq \# \text{cols of } A$.

**Lem 4** $\mu$-incoherent $\Rightarrow$ $(s, \delta)\text{-RIP}$ for $\delta \leq 2s\mu$.

**Proof** Assume wolog, $\|x\|_2 = 1$ and $\|x\|_0 = s$.

$\|Ax\|^2_2 = \sum_i x(i)^2 a_i^T a_i + \sum_{i \neq j} x(i)x(j) a_i^T a_j$

$= 1 + \sum_{i \neq j} x(i)x(j) a_i^T a_j$

$|\sum_{i \neq j} x(i)x(j) a_i^T a_j| \leq \mu \sum_{i \neq j} |x(i)x(j)|$
\[
\mu \sum_{i \in j} |x(i) \cdot y(i)| \\
\leq \mu \sum_{i \in j} (x_i^2 + y_i^2) \\
\leq 2smu
\]

\[
\Rightarrow \|A \times b\|_2^2 \leq 1 + 2smu.
\]

and \[
\|A \times b\|_2^2 \geq 1 - 2smu.
\]

For random A, we can get much better parameters by a direct argument.