Today: The following are NP-hard:

- Maximizing a convex function over a convex set.
- Finding the point of largest norm in a polytope \((Ax \leq b)\)
- Computing a matrix \(p\)-norm, \(\|M\|_p, 2 \leq p \leq \infty\)

All will follow from reductions from Max-Cut.

This is a graph problem.

Recall that a graph \(G\), usually written \(G = (V, E)\), consists of a set of vertices (aka "nodes"), \(V\), and edges \(E\) connecting pairs of nodes.

Each edge is a subset of \(V\) of size 2, but we write the pair in parentheses, like \((a, b)\)

**Ex1** Social network

**Ex2**

A "cut" is a subset of the vertices, \(S \subseteq V\).

We count the size of the boundary of \(S\), \(\partial(S) = \) edges from inside to outside \(S\)

\[
S = \{a, d, b\} \\
\partial(S) = 3 \\
S = \{d\} \\
\partial(S) = 3 \\
S = \{b, d\} \\
\partial(S) = 4
\]
The maximum cut is \( \text{maxcut}(G) = \max_{S \subseteq V} |\partial(S)| \).

To make it into a decision problem, we give \( G \) and \( k \) as input.

\[ (G, k) \in \text{MaxCut} \text{ if } \exists S \subseteq V \text{ s.t. } |\partial(S)| \geq k \]

Answer is "yes" if \( G \) has a cut of size \( \geq k \).

\text{MaxCut is in NP}: S is the witness

And, it is \( \text{NP-complete} \).

One can not even approximate its value.

Hastad '01 proved is \( \text{NP-hard to even approximate within a factor } \frac{18}{17}. \)

That is, for every problem \( Y \in \text{NP}, \)

\( \exists \text{ a polynomial time algorithm } A \text{ that outputs } G, k \text{ s.t. } q \in Y \Rightarrow G \text{ has a cut of size } \geq k \)

\( q \notin Y \Rightarrow \text{ all cuts in } G \text{ are smaller than } \frac{17}{18} k \)

The result is a little tighter than this: \( \frac{17}{16} + \varepsilon \)

Is a very deep and complicated result building on a decade of research by many luminaries.
Degree of a vertex is \# of attached edges: 
\[ d_a = | \{ b : (a,b) \in E \} | \]

Degree 3-graphs: Berman & Karpinski '02
Reduce to graphs in which every vertex has degree 3
and
\[ q \in \mathcal{Y} \implies \text{maxcut}(G) \geq k \]
\[ q \notin \mathcal{Y} \implies \text{maxcut}(G) \leq \frac{332}{333} k = (1 - \frac{1}{333}) k \]
Will use this for p-norms.

---

Maximizing convex functions over convex sets

The maximum will be on the boundary, and values inside won't be very helpful.

Claim: the maximum will be at a "corner".
Def: $x$ is an extreme point of a convex set $C$ if there do not exist $y, z \in C$ and $\lambda \in (0, 1)$ s.t.
$$x = \lambda y + (1-\lambda)z.$$ 

Equivalently, for $\delta \neq 0$, $x + \delta \in C \Rightarrow x - \delta \notin C.$

When $C$ is a polytope like $Ax \leq b$, the extreme points are the corners.
Idea of proof for \( \{ x : \| x \|_\infty \leq 1 \} \)

Let \( y = (1, x(2), \ldots, x(n)) \) \( z = (-1, x(2), \ldots, x(n)) \)
\( x \in \overline{yz} \)

Convexity \( \Rightarrow \) \( \max(f(y), f(z)) \geq f(x) \), \( z \)

So, go to one of \( y \) or \( z \).

Repeat for each coordinate

Eventually hit a corner

Proof (for polytopes) let \( x_* \) be the maximum.

By Carathéodory's theorem,
\[ \exists \text{ corners } v_0, \ldots, v_d \text{ s.t. } x_* = \sum_{i=0}^{d} \lambda_i v_i \]

\( \lambda_i \geq 0, \sum \lambda_i = 1 \)

Convexity \( \Rightarrow f(x_*) \leq \sum_{i=0}^{d} \lambda_i f(v_i) \)

\( \Rightarrow \exists v_i \text{ s.t. } f(v_i) \geq f(x_*) \).
A convex relaxation of maxcut.

First, need to write algebraically.

Let $n = |V|$. For a set $S$, let $x_S(a) = \begin{cases} 1 & a \in S \\ -1 & a \notin S \end{cases}$

Then, for $(a,b) \in E$

$$|x_S(a) - x_S(b)| = \begin{cases} 2 & \text{if } (a,b) \in E(S) \\ 0 & \text{otherwise} \end{cases}$$

So, $\frac{1}{4} \sum_{(a,b) \in E} (x_S(a) - x_S(b))^2 = |E(S)|$

Let $Q(x) = \frac{1}{4} \sum_{(a,b) \in E} (x(a) - x(b))^2$

Is a sum of squares, so it is convex.

Claim $\max_{\|x\|_2 \leq 1} Q(x) = \maxcut(G)$

So, have a polynomial time reduction from maxcut to problem of maximizing $Q$ over $\|x\|_2 \leq 1$, a convex set.

$\Rightarrow$ maximizing convex quadratics over $\|x\|_2 \leq 1$

is NP hard.

The decision problem is: given $Q, t$, $\exists x: \|x\|_2 \leq 1$, $Q(x) \geq t$?

Is in NP.
Proof: If $S_x$ maximizes $|\Delta(S)|$, $Q(X_{S_x}) = \text{maxcut}(G)$.
So, $\max Q(X) \geq \text{maxcut}(G)$.
The maximum $X$ is at a corner, and so
$X_x \in \{\pm 1 \}^n \Rightarrow X_x = X_s$ for some $S$ 
And so, $Q(X_x) = Q(X_s) = |\Delta(S)| \leq \text{maxcut}(G)$

Note: Turned arbitrary maxcut problem into special convex quadratic.
If don't expect to solve maxcut, shouldn't expect to maximize convex quadratics.

It is a very special quadratic: the Laplacian
$Q(x) = \frac{1}{2}x^T L x$ where $L(a,b) = \begin{cases} \text{degree of } a \text{ if } a=b \\ -1 \text{ if } (a,b) \in E \\ 0 \text{ otherwise} \end{cases}$

$(L x)(a) = d_a x(a) - \sum_{b : (a,b) \in E} x(a) x(b)$

So, $x^T L x = \sum_a d_a x(a)^2 - \sum_a \sum_{b : (a,b) \in E} x(a) x(b)$

$= \sum_{(a,b) \in E} (x(a)^2 + x(b)^2) - 2 \sum_{(a,b) \in E} x(a) x(b)$

$= \sum_{(a,b) \in E} (x(a) - x(b))^2$
The problem \( \max_x \|x\|_2^2 \) s.t. \( Ax \leq b \) is \( \text{NP-hard} \).

Proof: We will reduce the problem
\[ (1) \quad \max_x x^T L x \text{ s.t. } \|x\|_2 \leq 1 \text{ to this one} \]

Consider \( x^T (L + I)x = x^T L x + x^T x \)
\[ (2) \quad \max_x x^T (L + I)x \text{ s.t. } \|x\|_2 \leq 1 = (1) + \eta \]

So, (2) is \( \text{NP-hard} \).

\( L + I \) is positive definite.
So, \( I \) invertible, symmetric \( B \) s.t. \( B^T B = L + I \).
\[ x^T (L + I)x = x^T B^T B x \]
Set \( y = Bx \), \( x = B^{-1} y \)

Then, \( (2) = \max_y y^T y \text{ s.t. } \|By\|_2 \leq 1 \)
\[ \|By\|_2 \leq 1 \iff \|b_i^T y\| \leq 1 \text{ and } -\|b_i^T y\| \leq 1 \]
where \( b_i \) are rows of \( B \).
So, can express \( \|By\|_2 \leq 1 \) in form \( Ax \leq b \).
**Matrix Norms for** $p > 2$.

\[ \|M\|_p = \max_x \frac{\|Mx\|_p}{\|x\|_p} = \max_{\|x\|_p = 1} \|Mx\|_p \]

- $p = 2$ is max singular value.
- $p = \infty$ is $\max_i \|M(i,:)\|_1$.
- $p = 1$ is $\max_i \|M(:,i)\|_1$.

For all other $p$ is UP-hard. Bhaskara & Vijayaraghavan show is UP-hard to approximate within any constant.

Write $(M, t) \in \text{Matrix } p\text{-norm} \text{ if } \|M\|_p \geq t$

**Thm** For all $p > 2$, $c > 1$, and $Y \in \text{NP}$, 3 poly time algorithm $A$ that outputs $M, t$ s.t.
- $q \in Y \Rightarrow \|M\|_p \geq ct$
- $q \notin Y \Rightarrow \|M\|_p \leq t$

**Note**: $A$ depends on $c$.

$\|M\|_p$ is convex in $M$. But, it is hard to evaluate.
How prove this for all $c > 1$?
First show for some $c > 1$, and then amplify.

Recall the Kronecker product:

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ \vdots & \ddots & \ddots & \vdots \\ a_{m1} B & \cdots & \cdots & a_{mn} B \end{pmatrix}$$

**Thm** $\| A \otimes B \|_p = \| A \|_p \cdot \| B \|_p$.

**Proof Sketch:**
1. For vectors $x$ and $y$,
   
   $\| x \otimes y \|_p = \| x \|_p \cdot \| y \|_p$

   and $(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$

   $\Rightarrow \| A \otimes B \|_p = \| A \|_p \cdot \| B \|_p$

2. $A \otimes B = (A \otimes I)(I \otimes B)$,
   
   so $\| A \otimes B \|_p \leq \| A \|_p \cdot \| I \otimes B \|_p = \| A \|_p \cdot \| B \|_p$

   Think about $I \otimes B = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & B \end{pmatrix}$
So, it is hard to distinguish $|M_1|_p \leq 5$ from $\geq 6$

is hard to distinguish $|M \otimes M_1|_p \leq 5^2$ from $\geq 6^2$

This squares the inapproximability factor.
Can do this any constant number of times, and still be a polynomial time reduction.

To get some constant:

Idea: want $\sum_{a \in \{0,1\}} |x(a) - x(b)|^p$

But, can’t force $x(a) \in \{\pm 1\}$.

So, add terms to penalize $|x(a)| \notin (1-\delta, 1+\delta)$

Claim 1: let $F(x) = \frac{|x+1|^p + |x-1|^p}{1 + |x|^p}$

a. $F(\pm 1) = 2^{p-1}$
b. $\forall x, \ F(x) \leq 2^{p-1}$
c. $\exists \delta > 0 \exists > 0$ s.t. $|x| \notin (1-\delta, 1+\delta)$

$\Rightarrow F(x) \leq 2^{p-1} - \delta$
Claim 2: \( \forall \varepsilon > 0 \exists C > 0 \text{ s.t.} \)

\[
F(x, y) = \frac{|x - y|^p + C \left[ |x + 1|^p + |x - 1|^p + |y + 1|^p + |y - 1|^p \right]}{2 + |x|^p + |y|^p}
\]

if \( x, y < 0 \) and \( x, y \in (1 - \varepsilon, 1 + \varepsilon) \)

\[
F(x, y) \leq C 2^{p-1} + \frac{(1 + |x|^p)2^{p-1}}{2 + |x|^p + |y|^p}
\]

if \( x, y > 0 \), \( x \in (1 - \varepsilon, 1 + \varepsilon) \) or \( y \in (1 - \varepsilon, 1 + \varepsilon) \)

\[
F(x, y) \leq C 2^{p-1} + \frac{(2\varepsilon)^p 2^{p-1}}{2 + |x|^p + |y|^p}
\]

For a 3-regular graph \( G \), set

\[
g(x) = \sum_{\text{neigh}(x)} |x(a) - x(b)|^p + 3C \left[ |1 + x(a)|^p + |1 - x(a)|^p + |1 + y(a)|^p + |1 - y(a)|^p \right]
\]

\[
3n + 3 \sum_a |x(a)|^p
\]

Lemma: \( \max(\overline{G}) = |\overline{\overline{G}}| \leq \frac{3}{2}n \).

\[
g(x) = C 2^{p-1} + \varepsilon 2^{p-2}, \text{ and}
\]

\[
\forall x \quad g(x) \leq C 2^{p-1} + (2\varepsilon)^p 2^{p-2} + \varepsilon (1 + |x|^p + |y|^p) 2^{p-2}
\]

For any \( 0 < \delta_0 < \delta_1 \), is a small enough \( \varepsilon \) so that computing \( \max_x g(x) \) allows one to distinguish

\[
\max(\overline{G}) \leq \delta_0 \frac{3}{2}n \text{ from } \geq \delta_1 \frac{3}{2}n
\]
So, is \( \text{NP-hard to compute } \max_x g(x) \) to within some constant.

To turn into a matrix, introduce new variable \( x(a) \), which want to have value \( n^{1/p} \) so \( x(a)/n^{1/p} = 1 \), and consider

\[
\sum_{(a,b) \in E} |x(a) - x(b)|^p \\
+ 3C \left[ \left| \frac{x(a)}{n^{1/p}} - x(a) \right|^p \right. \\
\left. + \left| \frac{x(a)}{n^{1/p}} + x(a) \right|^p \right. \\
\left. + \left| \frac{x(b)}{n^{1/p}} - x(b) \right|^p \right. \\
\left. + \left| \frac{x(b)}{n^{1/p}} + x(b) \right|^p \right] \\
\frac{3|x(a)|^p + 3 \sum_a |x(a)|^p}{3|x(a)|^p + 3 \sum_a |x(a)|^p}
\]

Show its max value is same as max of \( g \).

Denominator = \( 3 \| x \|_{1/p}^p \) and

Numerator = \( \| Mx \|_{1/p}^p \) for some matrix \( M \).

So, is hard to compute \( \max_x \frac{\| Mx \|_{1/p}^p}{\| x \|_{1/p}^p} = \| M \|_{1/p}^p \),

and to approximate it up to some constant.