Today: The following are NP-hard:

Maximizing a convex function over a convex set.

Finding the point of largest norm in a polytope ($Ax \le b$)

Computing a matrix p-norm, $\|M\|_p$, $2 < p < \infty$

All will follow from reductions from Max-Cut.

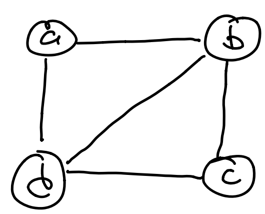
This is a graph problem.

Recall that a graph G, usually written $G = (V, E)$,
consists of a set of vertices (aka "nodes"), V,
and edges E connecting pairs of nodes.

Each edge is a subset of V of size 2, but
we write the pair in parentheses, like (a, b)

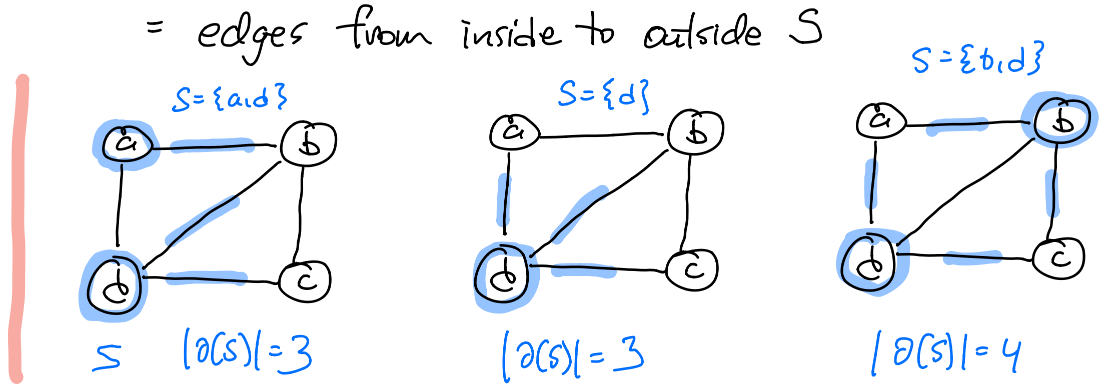
Ex 1 Social network

Ex 2



A "cut" is a subset of the vertices, $S \subset V$.

We count the size of the boundary of S, $\partial(S)$
= edges from inside to outside S



The maximum cut is $\text{maxcut}(G) = \max_{S \subset V} |\partial(S)|$

To make into a decision problem, we give G and k as input.

$(G, k) \in \text{MaxCut}$ if $\exists\, S \subset V$ s.t. $|\partial(S)| \geq k$

answer is "yes" if G has a cut of size $\geq k$

MaxCut is in NP: S is the witness
And, it is NP-complete.

One can not even approximate its value.
Håstad '01 proved is NP-hard to even approximate within a factor $18/17$.
That is, for every problem $Y \in NP$,
$\exists$ a polynomial time algorithm A that outputs G, k
s.t. $q \in Y \Rightarrow G$ has a cut of size $\geq k$
$q \notin Y \Rightarrow$ all cuts in G are smaller than $\frac{17}{18} k$

The result is a little tighter than this: $\frac{17}{16} + \varepsilon$
Is a very deep and complicated result building on a decade of research by many luminaries.

Degree of a vertex is # of attached edges:

$$d_a = |\{b : (a,b) \in E\}|$$

Degree 3-Graphs: Berman & Karpinski '02

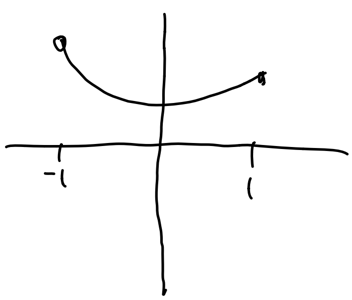
reduce to graphs in which every vertex has degree 3 and

$$q \in Y \Rightarrow \text{maxcut}(G) \geq k$$

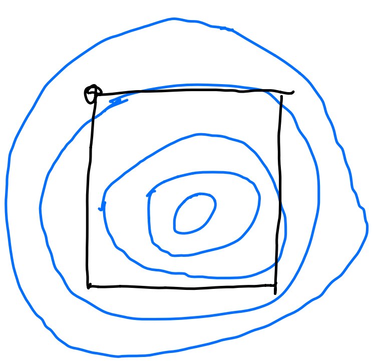
$$q \notin Y \Rightarrow \text{maxcut}(G) \leq \frac{332}{333} k = \left(1 - \frac{1}{333}\right) k$$

Will use this for p-norms.

Maximizing convex functions over convex sets



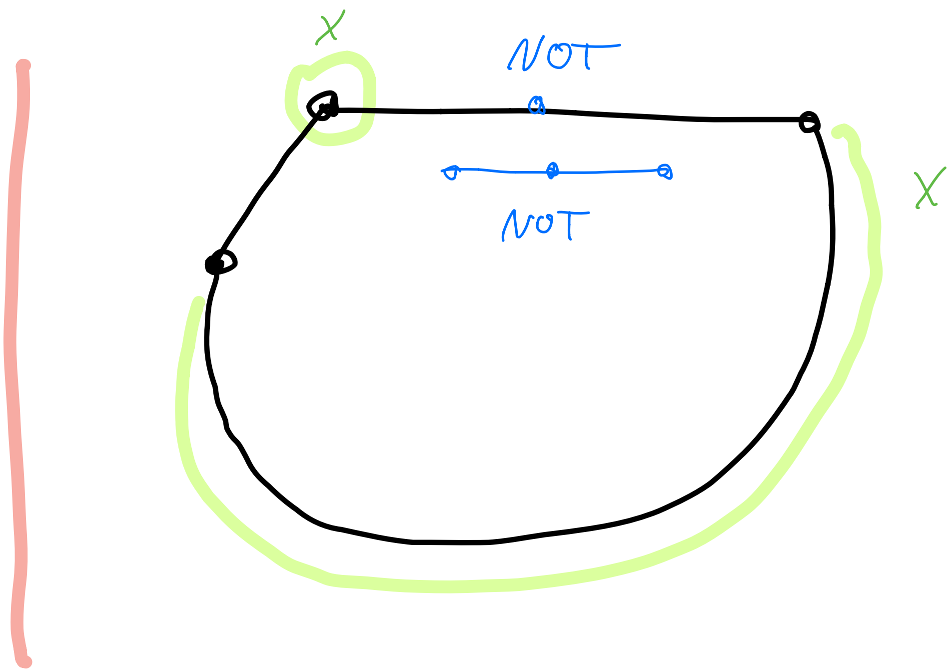
the maximum will be on the boundary, and values inside won't be very helpful.



Claim: the maximum will be at a "corner".

Def x is an <u>extreme point</u> of a convex set C if there do NOT exist $y, z \in C$ and $\lambda \in (0,1)$ s.t.

$$x = \lambda y + (1-\lambda) z.$$



Equivalently, for $\delta \neq 0$, $x + \delta \in C \Rightarrow x - \delta \notin C$.

When C is a polytope like $Ax \le b$, the extreme points are the corners.

<u>Idea of proof</u> for $\{x : \|x\|_\infty \le 1\}$

let $y = (1, x(2), \dots, x(n))$ $\quad z = (-1, x(2), \dots, x(n))$

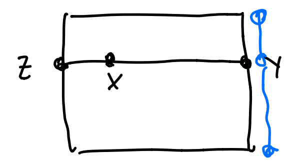
$x \in \overline{yz}$

Convexity $\Rightarrow \max(f(y), f(z)) \ge f(x)$.

So, go to one of y ad z.

Repeat for each coordinate

Eventually hit a corner



<u>Proof.</u> (for polytopes) let x_* be the maximum.

By Caratheodory's theorem,

$\exists$ corners $v_0, \dots, v_d$ s.t. $x_* = \sum_{i=0}^{d} \lambda_i v_i$

$\lambda_i \ge 0$, $\sum \lambda_i = 1$

Convexity $\Rightarrow f(x_*) \le \sum_{i=0}^{d} \lambda_i f(v_i)$

$\Rightarrow \exists v_i$ s.t. $f(v_i) \ge f(x_*)$.

A convex relaxation of maxcut.

First, need to write algebraically.

Let $n = |V|$. For a set S, let $x_S(a) = \begin{cases} 1 & a \in S \\ -1 & a \notin S \end{cases}$

Then, for $(a,b) \in E$

$$|x_S(a) - x_S(b)| = \begin{cases} 2 & \text{if } (a,b) \in \partial(S) \quad \left(\begin{array}{l} a \in S,\ b \notin S \\ \text{or } a \notin S,\ b \in S \end{array}\right. \\ 0 & \text{o.w.} \end{cases}$$

So, $\frac{1}{4} \sum_{(a,b) \in E} (x_S(a) - x_S(b))^2 = |\partial(S)|$

Let $Q(x) = \frac{1}{4} \sum_{(a,b) \in E} (x(a) - x(b))^2$

Is a sum of squares, so it is convex.

<u>Claim</u> $\max_{\|x\|_\infty \le 1} Q(x) = \text{maxcut}(G)$

So, have a polynomial time reduction from maxcut to problem of maximizing Q over $\|x\|_\infty \le 1$, a convex set.

$\Rightarrow$ maximizing convex quadratics over $\|x\|_\infty \le 1$ is NP hard.

The decision problem is: given Q, t, $\exists x : \|x\|_\infty \le 1$, $Q(x) \ge t$?
Is in NP.

<u>Proof</u> If S_* maximizes $|\partial(S)|$, $Q(x_{S_*}) = \text{maxcut}(G)$

So, $\max Q(x) \geq \text{maxcut}(G)$.

The maximum x_* is at a corner, and so

$x_* \in \{\pm 1\}^n \Rightarrow x_* = x_S$ for some S

And so, $Q(x_*) = Q(x_S) = |\partial(S)| \leq \text{maxcut}(G)$

Note: Turned arbitrary maxcut problem into special convex quadratic.

If don't expect to solve maxcut, shouldn't expect to maximize convex quadratics.

It is a very special quadratic: the Laplacian

$$Q(x) = \frac{1}{4} x^T L x \quad \text{where} \quad L(a,b) = \begin{cases} \text{degree of } a & \text{if } a=b \\ -1 & \text{if } (a,b) \in E \\ 0 & \text{o.w.} \end{cases}$$

$$(Lx)(a) = d_a x(a) - \sum_{b:(a,b)\in E} x(a) x(b)$$

$$\text{So,} \quad x^T L x = \sum_a d_a x(a)^2 - \sum_a \sum_{b:(a,b)\in E} x(a) x(b)$$

$$= \sum_{(a,b)\in E} \left(x(a)^2 + x(b)^2\right) - 2 \sum_{(a,b)\in E} x(a) x(b)$$

$$= \sum_{(a,b)\in E} \left(x(a) - x(b)\right)^2$$

Thm The problem $\max_x \|x\|_2^2$ s.t. $Ax \le b$ is NP-hard

proof We will reduce the problem

(1) $\max_x \; x^T L x$ s.t. $\|x\|_\infty \le 1$ to this one

Consider $x^T (L+I) x = x^T L x + x^T x$

(2) $\max_x \; x^T (L+I) x$ s.t. $\|x\|_\infty \le 1 \;=\; (1) + n$

So, (2) is NP-hard

$L+I$ is positive definite.

So, $\exists$ invertible, symmetric B s.t. $B^T B = L+I$.

$x^T (L+I) x = x^T B^T B x$

Set $y = Bx$, $x = B^{-1} y$

Then, $(2) = \max_y \; y^T y$ s.t. $\|By\|_\infty \le 1$

$\|By\|_\infty \le 1 \iff b_i^T y \le 1$ and $-b_i^T y \le 1$

where $b_1 \dots b_n$ are rows of B.

So, can express $\|By\|_\infty \le 1$ in form $Ax \le b$.

Matrix Norms for $p > 2$.

$$\|M\|_p = \max_x \frac{\|Mx\|_p}{\|x\|_p} = \max_{\|x\|_p \le 1} \|Mx\|_p$$

$p=2$ is max singular value

$p=\infty$ is $\max_i \|M(i,\cdot)\|_1$

$p=1$ is $\max_i \|M(\cdot,i)\|_1$

For all other p is NP-hard.

after we correct minor mistakes

Bhaskara & Vijayaraghavan show is NP-hard to approximate within any constant.

Write $(M,t) \in$ Matrix p-norm if $\|M\|_p \ge t$

Thm For all $p>2$, $c>1$, and $Y \in NP$,

$\exists$ poly time algorithm A that outputs M, t s.t.

$q \in Y \Rightarrow \|M\|_p \ge ct$

$q \notin Y \Rightarrow \|M\|_p \le t$

Note: A depends on c.

$\|M\|_p$ is convex in M. But, it is hard to evaluate.

How prove this for all $c > 1$?
First show for some $c > 1$, and then amplify.

Recall the Kronecker product:

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ \vdots & & & \\ a_{n1} B & \cdots & & a_{nn} B \end{pmatrix}$$

<u>Thm</u> $\|A \otimes B\|_p = \|A\|_p \cdot \|B\|_p$.

proof sketch:

1. for vectors x and y,
   $\|x \otimes y\|_p = \|x\|_p \cdot \|y\|_p$
   and $(A \otimes B)(x \otimes y) = (Ax) \otimes (By)$

   $\Rightarrow \|A \otimes B\|_p \geq \|A\|_p \|B\|_p$

2. $A \otimes B = (A \otimes I)(I \otimes B)$,
   so $\|A \otimes B\|_p \leq \|A \otimes I\|_p \|I \otimes B\|_p = \|A\|_p \|B\|_p$

   think about $I \otimes B = \begin{pmatrix} B & 0 & 0 & & \\ 0 & B & 0 & & \\ 0 & 0 & B & & \\ & & & \ddots & \\ & & & & B \end{pmatrix}$

So, if is hard to distinguish $\|M\|_p \le s$ from $\ge b$
is hard to distinguish $\|M \otimes M\|_p \le s^2$ from $\ge b^2$

This squares the inapproximability factor.
Can do this any constant number of times,
and still be a polynomial time reduction.

To get some constant:

Idea: want $\sum_{(a,b)\in E} |x(a) - x(b)|^p$

But, can't force $x(a) \in \pm 1$.

So, add terms to penalize $|x(a)| \notin (1-\varepsilon, 1+\varepsilon)$

<u>Claim 1</u>: let $F(x) = \dfrac{|x+1|^p + |x-1|^p}{1 + |x|^p}$

a. $F(\pm 1) = 2^{p-1}$
b. $\forall x,\ F(x) \le 2^{p-1}$
c. $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|x| \notin (1-\varepsilon, 1+\varepsilon)$
$\Rightarrow F(x) \le 2^{p-1} - \delta$

Claim 2: $\forall\, \varepsilon > 0 \;\exists\, C > 0$ s.t.

for $$F(x,y) = \frac{|x-y|^p + C\left[|x+1|^p + |x-1|^p + |y+1|^p + |y-1|^p\right]}{2 + |x|^p + |y|^p}$$

if $xy < 0$ and $x, y \in (1-\varepsilon, 1+\varepsilon)$
$$F(x,y) \le C\, 2^{p-1} + \frac{(1+\varepsilon)^p\, 2^{p-1}}{2 + |x|^p + |y|^p}$$

if $xy \ge 0$, $x \notin (1-\varepsilon, 1+\varepsilon)$ or $y \notin (1-\varepsilon, 1+\varepsilon)$

$$F(x,y) \le C\, 2^{p-1} + \frac{(2\varepsilon)^p\, 2^{p-1}}{2 + |x|^p + |y|^p}$$

For a 3-regular graph G, set

$$g(x) = \frac{\sum_{(a,b)\in E} |x(a) - x(b)|^p + 3C\left[|1+x(a)|^p + |1-x(a)|^p + |1+y(a)|^p + |1-y(a)|^p\right]}{3n + 3\sum_a |x(a)|^p}$$

<u>Lem</u> Let $\mathrm{maxcut}(G) = |\partial(S)| = \gamma \frac{3}{2} n$.

$g(X_S) = C\, 2^{p-1} + \gamma\, 2^{p-2}$, and

$\forall x \;\; g(x) \le C\, 2^{p-1} + (2\varepsilon)^p 2^{p-2} + \gamma (1+\varepsilon)^p (1-\varepsilon)^p 2^{p-2}$

For any $0 < \gamma_0 < \gamma_1$, is a small enough ε so that computing $\max_x g(x)$ allows one to distinguish

$\mathrm{maxcut}(G) \le \gamma_0 \frac{3}{2} n$ from $\ge \gamma_1 \frac{3}{2} n$

So, is NP-hard to compute $\max_x g(x)$ to within some constant

To turn into a matrix, introduce new variable $x(0)$, which want to have value $n^{1/p}$ so $x(0)/n^{1/p} = 1$, and consider

$$\frac{\displaystyle\sum_{(a,b)\in E} |x(a) - x(b)|^p + 3C\left[\left|\frac{x(0)}{n^{1/p}} - x(a)\right|^p + \left|\frac{x(0)}{n^{1/p}} + x(a)\right|^p + \left|\frac{x(0)}{n^{1/p}} - x(b)\right|^p + \left|\frac{x(0)}{n^{1/p}} - x(b)\right|^p\right]}{3|x(0)|^p + 3\sum_a |x(a)|^p}$$

Show its max value is same as max of g.

Denominator $= 3\|x\|_p^p$ and
Numerator $= \|Mx\|_p^p$ for some matrix M.

So, is hard to compute $\displaystyle\max_x \frac{\|Mx\|_p}{\|x\|_p} = \|M\|_p$,

and to approximate it up to some constant.