Last lecture we considered the program
\[ \begin{align*}
\text{max } & \alpha \\
\text{s.t. } & \alpha \in CH(\alpha_1, \ldots, \alpha_m) \\
\text{and we assumed } & 0 \in CH(\alpha_1, \ldots, \alpha_m)
\end{align*} \tag{P1} \]

We considered the dual program
\[ \begin{align*}
\text{max } & c^T \gamma \\
\text{s.t. } & \alpha_i^T \gamma \leq 1, \quad \forall i
\end{align*} \tag{D1} \]

And proved that for optimal \( \alpha_\star \) and \( \gamma_\star \),
\[ \alpha_\star = \frac{1}{C^T \gamma_\star} \]

We will now generalize these.
First, \( \alpha \in CH(\alpha_1, \ldots, \alpha_m) \) iff \( \exists t \geq 0, \quad 1^T t = 1 \) s.t.
\[ \alpha = A t, \quad \text{where } A = (\alpha_1, \ldots, \alpha_m) \]

If we set \( x(i) = t(i)/\alpha_i \), \( 1^T x = 1/\alpha \) so
larger \( \alpha \) corresponds to smaller \( 1^T x \), and it is equivalent to solve
\[ \begin{align*}
\text{min } & 1^T x \\
\text{s.t. } & Ax = c, \quad x \geq 0
\end{align*} \tag{P2} \]
For the optimal $x^*$, we now have

$$C^T y^* = 1^T x^*$$

So, solutions to $P_2$ upper bound solutions to $D_1$.

But, what if $0 \notin \CH(a,...,a^m)$?
Or, what if there is no $x$ s.t. $Ax=c, x \geq 0$.
Farkas' lemma

says that $\exists \gamma$ s.t. $a_i^T \gamma \leq 0 \ \forall i$
and $c^T \gamma > 0$

That is, value of $D_1 = \infty$, by considering $\mu_i, \mu \rightarrow \infty$.

Warm up: consider the problem $Ax=c$, no assumptions on $A$.
either there is a solution, or
there exists $\gamma$ s.t. $\gamma^T A = 0$, $\gamma^T c = 1$

This is strengthened by

Farkas' lemma  exactly one of the following holds
i. $\exists x \geq 0$ s.t. $Ax=c$
ii. $\exists \gamma$ s.t. $A^T \gamma = 0$, $c^T \gamma < 0$
Proof: If \( \exists y \) s.t. \( A^T y \leq 0, \ C^T y < 0 \)
then is no solution to \( i \) because
\[ Ax = c \Rightarrow x^T A^T y = c^T y < 0 \]
But, \( x \geq 0 \) and \( A^Ty = 0 \Rightarrow x^T A^T y = 0 \),
a contradiction.

To go the other way, note that
\( \exists Ax: x \geq 0 \) is a convex set.

So, if \( C \notin \exists Ax: x \geq 0 \), there is a
separating hyper-plane: \( y \) s.t.
\( y^T C < y^T Ax \) for all \( x \geq 0 \).

As \( x = 0 \) is possible, \( y^T C < 0 \)
To see \( y^T Ax = 0 \ \forall x \), not that if \( y^T Ax < 0 \),
then \( y^T A(x) \to 00 \) as \( x \to 0 \).

So, \( y^T Ax = 0 \ \forall x \)

Conclusion: P2 is feasible iff D1 is bounded.
D1 is always feasible: \( y = 0 \).

There are other variants like
Exactly one of these holds:
1. \( \exists y \) s.t. \( A y \leq b \)
2. \( \exists x \) s.t. \( A^T x = 0, \ x^T b = -1, \ x \geq 0 \)
The general standard LPs replace 1 with an arbitrary \( b \), and are

\[
P_3: \quad \min b^T x \\
\text{st. } A x = c, \quad x \geq 0
\]

\[
D_3: \quad \max c^T y \\
\text{st. } A^T y \leq b
\]

We will see that these are dual to each other.

Weak duality is \( c^T y \leq b^T x \)

\[\text{Proof: } c^T y = y^T c = y^T A x = b^T x, \text{ because } x \geq 0.\]

The complications are that there might not be \( x \) or \( y \) satisfying the conditions of \( P_3 \) or \( D_3 \), and the values of \( P_3 \) and \( D_3 \) can be \(-\infty\) or \( \infty \).

To see why, consider the geometry
Geometric view of D:

\( A^T y \leq b \) is \( \cap \) halfspaces

MAT says to go as far as possible in one direction.

Example: \( y_2 \geq 1, \ y_1 \leq 1 - y_2 \leq 1 \)

\[
\begin{align*}
    \text{if } c^T y &= y_1 - y_2, & c &= (1, -1) \\
    y^* &= (1, 1) \\
    \text{if } c^T y &= y_2 - y_1 \\
    \text{is unbounded: consider } (0, y_2) & \quad y_2 \to \infty
\end{align*}
\]

\( A x = c, \ x \geq 0 \) also looks like this.

P3 is feasible if \( \exists x \text{ st. } A x = c, \ x \geq 0 \)

D3 is feasible if \( \exists y \text{ st. } A^T y \leq b \).

They are infeasible when \( x \) or \( y \) do not exist.

Weak duality tells us that:

\[
\begin{align*}
    \text{value } (P3) &= -\infty \implies D3 \text{ is infeasible, and} \\
    \text{value } (D3) &= +\infty \implies P3 \text{ is infeasible.}
\end{align*}
\]

Strong Duality Theorem:

P3 feasible & bounded \( \iff \) D3 feasible & bounded,

in which case have same value.
We prove most of this.

First, it holds if \( b > 0 \).

**Proof.** Recall \( A = (a_1, \ldots, a_m) \).

Set \( \hat{x}(i) = x(i)b(i) \), so \( b^T x = \hat{x}^T \).

and set \( \hat{a}_i = \frac{x(i)}{b(i)} a_i \), so \( \hat{A}^x = Ax \).

So, \( P_3 \) now has form \( P_2 \).

And, \( \hat{A}^y = (\hat{a}_i y)_i = (\frac{a_i^T y}{b(i)} y)_i \leq 1 \iff \hat{A}^y \leq b \),

because \( b > 0 \)

So, \( D_3 \) now has form \( D_1 \).

and we know that \( P_2 \) and \( D_1 \) satisfy strong duality.

Consider general \( b \), and \( D_3 \) strictly feasible.

That is \( \exists \ y_0 \ s.t. \ \hat{A}^y_0 \leq b \).

Set \( \hat{y} = y - y_0 \), \( \hat{b} = b - \hat{A}^y_0 \).

So, \( \hat{b} > 0 \)

And, \( \hat{A}^\hat{y} \leq \hat{b} \iff \hat{A}^T(y - y_0) \leq b - \hat{A}^y_0 \iff \hat{A}^\hat{y} \leq b \).

\[ c^T \hat{y} = c^T(y - y_0) = c^T \hat{y} - c^T y_0. \]

And, \( \hat{b}^T x = \hat{b}^T x - y_0^T A x = \hat{b}^T x - y_0^T c = \hat{b}^T x - c^T y_0 \)

So, \( c^T \hat{y} = b^T x \iff c^T \hat{y} = b^T x \)

and the solutions are the same.
What if \( \exists y_0: \mathbf{A}^T y_0 \leq \mathbf{b} \) but does not exist \( \mathbf{A}^T y_0 < \mathbf{b} \)?

Then \( \exists \ i \ s.t. \ \mathbf{A}^T y_0 \leq \mathbf{b} \Rightarrow \mathbf{A}^T y_0 \ = \ \mathbf{b}(i), \)

so \( y_0 \) is restricted to a subspace.

Will handle this in homework.

How to tell if LP is feasible & find \( y \):

To find \( y \ s.t. \ \mathbf{A}^T y \leq \mathbf{b}, \)

solve the program \( \min t \ s.t. \ \mathbf{A}^T y \leq \mathbf{b} + t \mathbf{1} \)

\( \Leftrightarrow \) \( \min t \ s.t. \ \mathbf{A}^T y + t \mathbf{1} \leq \mathbf{b} \)

We know is feasible for big \( t \) and \( y = \mathbf{1} \).

If can achieve \( t \leq 0 \), is feasible.

Can rewrite by choosing \( t \) s.t. \( b + t \mathbf{1} \geq 0 \),

And solving \( \min t \ s.t. \ \mathbf{A}^T y + t \mathbf{1} \leq b + t \mathbf{1} \)

The rhs, \( b + t \mathbf{1} \) is now positive.

Is called a Phase I problem.

Then solve the original LP in Phase II.
Note: solving problems like \( \max c^T y \) s.t. \( A^T y \leq b \)

is essentially equivalent to testing feasibility.

First, ask if \( 3^T y \) s.t. \( A^T y \leq b \).
If so, ask if \( 3^T y \) s.t. \( A^T y \leq b \) and \( c^T y \geq 0 \) \((c^T y = 0)\)
now, play 20 questions...
if yes try \( c^T y = 2, c^T y = 3, c^T y = 4 \),
doubling until get a "no".
Then do binary search.

Both infeasible? \( Ax = c, x \geq 0, A^T y \leq b \)
consider \( A = \begin{bmatrix} 0 \end{bmatrix}, c = \begin{bmatrix} 1 \end{bmatrix}, b = \begin{bmatrix} 1 \end{bmatrix} \)
Complementary Slackness.

\[ \text{P: } \min b^T x \quad \text{D: } \max c^T y \]
\[ \text{st. } A x = c, \ x \geq 0 \quad \text{st. } A^T y \leq b \]

Put slack variables \( s \) into D:
\[ \max c^T y \quad \text{st. } A^T y + s = b, \ s \geq 0 \]

The duality gap is \( b^T x - c^T y = b^T x - x^T A^T y = x^T (b - A^T y) = x^T s \)

So, \( P = D \iff x^T s = 0 \).

Complementary Slackness Theorem

These are equivalent:

i. \( x^*_x, y^*_x \) and \( s^*_x \) are optimal
ii. \( x^*_x s^*_x = 0 \)
iii. \( x^*_x s^*_j = 0 \) for all \( j \)
iv. \( s^*_x c^*_j > 0 \implies x^*_x c^*_j = 0 \)

i \( \iff \) ii by duality gap arg.
i \( \iff \) iii by \( x^*_x, s^*_x \geq 0 \)
i \( \iff \) iv by logic, \( x^*_x s^*_x \geq 0 \)