Last lecture we considered the program

$$\max \alpha \quad \text{s.t. } \alpha c \in CH(a_1, \dots, a_m) \qquad (P1)$$

and we assumed $0 \in CH(a_1 \dots a_m)$

We considered the dual program

$$\max c^T y \quad \text{s.t. } a_i^T y \le 1, \ \forall i \qquad (D1)$$

And proved that for optimal α_* and y_*,

$$\alpha_* = 1/c^T y_*$$

We will now generalize these.

First, $\alpha c \in CH(a_1 \dots a_m)$ iff $\exists t \ge 0$, $1^T t = 1$ s.t.

$$\alpha c = At,$$

where $A = \begin{pmatrix} | & & | \\ a_1 & \dots & a_m \\ | & & | \end{pmatrix}$

If we set $x(i) = t(i)/\alpha$, $1^T x = 1/\alpha$ so larger α corresponds to smaller $1^T x$, and it is equivalent to solve

$$\min 1^T x \quad \text{s.t. } Ax = c, \ x \ge 0 \qquad (P2)$$

For the optimal x_*, we now have

$$c^T y_* = 1^T x_*$$

So, solutions to P2 upper bound solutions to D1.

But, what if $0 \notin CH(a_1 \dots a_m)$?

Or, what if there is no x s.t. $Ax = c$, $x \geq 0$.

Farkas' lemma

says that $\exists y$ s.t. $a_i^T y \leq 0 \ \forall i$

and $c^T y > 0$

That is, value of D1 $= \infty$, by considering μy, $\mu \to \infty$.

Warm up: consider the problem $Ax = c$, no assumptions on A.

either there is a solution, or

there exists y s.t. $y^T A = 0$, $y^T c = 1$

This is strengthened by

Farkas' lemma Exactly one of the following holds

i. $\exists x \geq 0$ s.t. $Ax = c$

ii. $\exists y$ s.t. $A^T y \geq 0$, $c^T y < 0$

<u>proof</u> If $\exists y$ s.t. $A^T y \geq 0$, $c^T y < 0$
then is no solution to i because
$Ax = c \Rightarrow x^T A^T y = c^T y < 0$
But, $x \geq 0$ and $A^T y \geq 0 \Rightarrow x^T A^T y \geq 0$,
a contradiction.

To go the other way, note that
$\{Ax : x \geq 0\}$ is a convex set.

So, if $c \notin \{Ax : x \geq 0\}$, there is a
separating hyperplane : y s.t.
$y^T c < y^T A x$ for all $x \geq 0$.
As $x = 0$ is possible, $y^T c < 0$
To see $y^T A x \geq 0 \ \forall x$, note that if $y^T A x < 0$,
then $y^T A(\lambda x) \to -\infty$ as $\lambda \to \infty$
So, $y^T A x \geq 0 \ \forall x$

Conclusion : P2 is feasible iff D1 is bounded.
D1 is always feasible : $y = \bar{0}$.

There are other variants like
Exactly one of these holds:
i. $\exists y$ s.t. $Ay \leq b$
ii $\exists x$ s.t. $A^T x = 0$, $x^T b = -1$, $x \geq 0$

The general standard LPs replace 1 with an arbitrary b, and are

P3: $\min b^T x$

s.t. $Ax = c$, $x \geq 0$

D3: $\max c^T y$

s.t. $A^T y \leq b$

We will see that these are dual to each other.

Weak duality is $c^T y \leq b^T x$

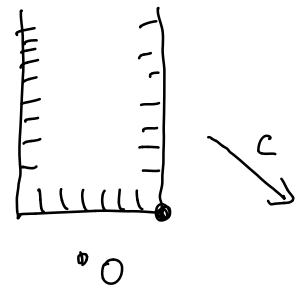
<u>Proof:</u> $c^T y = y^T c = y^T A x \leq b^T x$, because $x \geq 0$.

The complications are that there might not be x or y satisfying the conditions of P3 or D3, and the values of P3 and D3 can be $-\infty$ or ∞.

To see why, consider the geometry

Geometric view of D3: $A^T y \le b$ is $\bigcap$ halfspaces
max $c^T y$ says to go as far as possible in one direction.

<u>Example</u> $y_2 \ge 1$, $y_1 \le 1$, $-y_1 \le 1$



if $c^T y = y_1 - y_2$ $c = (1,-1)$
$y^* = (1,1)$

if $c^T y = y_2 - y_1$
is unbounded: consider $(0, y_2)$ $y_2 \to \infty$

$Ax = c$ $x \ge 0$ also looks like this.

P3 is feasible if $\exists x$ s.t. $Ax = c$, $x \ge 0$
D3 is feasible if $\exists y$ s.t. $A^T y \le b$.

They are <u>infeasible</u> when x or y do not exist.

Weak duality tells us that
value (P3) $= -\infty$ $\Rightarrow$ D3 is infeasible, and
value (D3) $= +\infty$ $\Rightarrow$ P3 is infeasible.

Strong Duality Theorem:
P3 feasible & bounded $\Leftrightarrow$ D3 feasible & bounded,
in which case have same value

We prove most of this.

First, it holds if $b > 0$.

<u>proof.</u> Recall $A = (a_1 \cdots a_m)$.

Set $\hat{x}(i) = x(i)\,b(i)$, so $b^T x = \mathbf{1}^T \hat{x}$

and set $\hat{a}_i = \frac{1}{b(i)} a_i$, so $\hat{A}\hat{x} = Ax$

So, P3 now has form P2.

And, $\hat{A}^T y = (\hat{a}_i^T y)_i = \left(\frac{a_i^T}{b(i)} y\right)_i \le 1$ iff $A^T y \le b$,

because $b > 0$

So, D3 now has form D1,

and we know that P2 and D1 satisfy strong duality.

Consider general b, and D3 strictly feasible.

That is $\exists\, y_0$ s.t. $A^T y_0 < b$.

Set $\hat{y} = y - y_0$. $\hat{b} = b - A^T y_0$. So $\hat{b} > 0$

And, $A^T \hat{y} \le \hat{b} \iff A^T(y - y_0) \le b - A^T y_0 \iff A^T y \le b$.

$c^T \hat{y} = c^T(y - y_0) = c^T y - c^T y_0$.

And, $\hat{b}^T x = b^T x - y_0^T A x = b^T x - y_0^T c = b^T x - c^T y_0$

So, $c^T y = b^T x \iff c^T \hat{y} = \hat{b}^T x$

and the solutions are the same.

What if $\exists y_0$: $A^T y_0 \le b$ but does not exist $A^T y_0 < b$?
Then $\exists i$ s.t. $A^T y_0 \le b \Rightarrow a_i^T y_0 = b(i)$,
so y_0 is restricted to a subspace.

Will handle this in homework.

How to tell if LP is feasible & find y:

To find y s.t. $A^T y \le b$,
solve the program $\min t$ s.t. $A^T y \le b + t\mathbb{1}$
$\iff \min t$ s.t. $A^T y + t\mathbb{1} \le b$
We know is feasible for big t and $y=1$.

If can achieve $t \le 0$, is feasible

Can rewrite by choosing t_0 s.t. $b + t_0\mathbb{1} \ge 0$,
And solving $\min t$ s.t. $A^T y + t\mathbb{1} \le b + t_0\mathbb{1}$
The rhs, $b + t_0\mathbb{1}$ is now positive.

Is called a Phase I problem.
Then solve the original LP in Phase II.

Note: solving problems like $\max c^T y$ s.t. $A^T y \le b$

is essentially equivalent to testing feasibility

First, ask if $\exists y$ s.t. $A^T y \le b$.

If so, ask if $\exists y$ s.t. $A^T y \le b$ and $c^T y \ge 0$ $\left((-c)^T y \le 0\right)$

now, play 20 questions...

if yes try $c^T y \ge 1$, $c^T y \ge 2$, $c^T y \ge 4$,

doubling until get a "no".

Then do binary search.

Both infeasible? $Ax = c$, $x \ge 0$ $A^T y \le b$

consider $A = \bar{0}$ $c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $b = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Complementary Slackness.

P: $\min b^T x$ s.t. $Ax = c,\ x \geq 0$

D: $\max c^T y$ s.t. $A^T y \leq b$

Put slack variables s into D:

$$\max c^T y \quad \text{s.t.} \quad A^T y + s = b, \quad s \geq 0$$

The duality gap is $b^T x - c^T y = b^T x - x^T A^T y$

$= x^T(b - A^T y) = x^T s$

So, $P = D \Rightarrow x^T s = 0$.

Complementary Slackness Theorem

These are equivalent:

i. x_*, y_* and s_* are optimal

ii. $x_*^T s_* = 0$

iii. $x'(j)\, s(j) = 0$ for all j

iv. $s_*(j) > 0 \Rightarrow x_*(j) = 0$

i $\Leftrightarrow$ ii by duality gap arg.

ii $\Leftrightarrow$ iii by $x_* \geq 0$, $s_* \geq 0$

iii $\Leftrightarrow$ iv by logic, $x, s \geq 0$