Dan's Favorite LP:

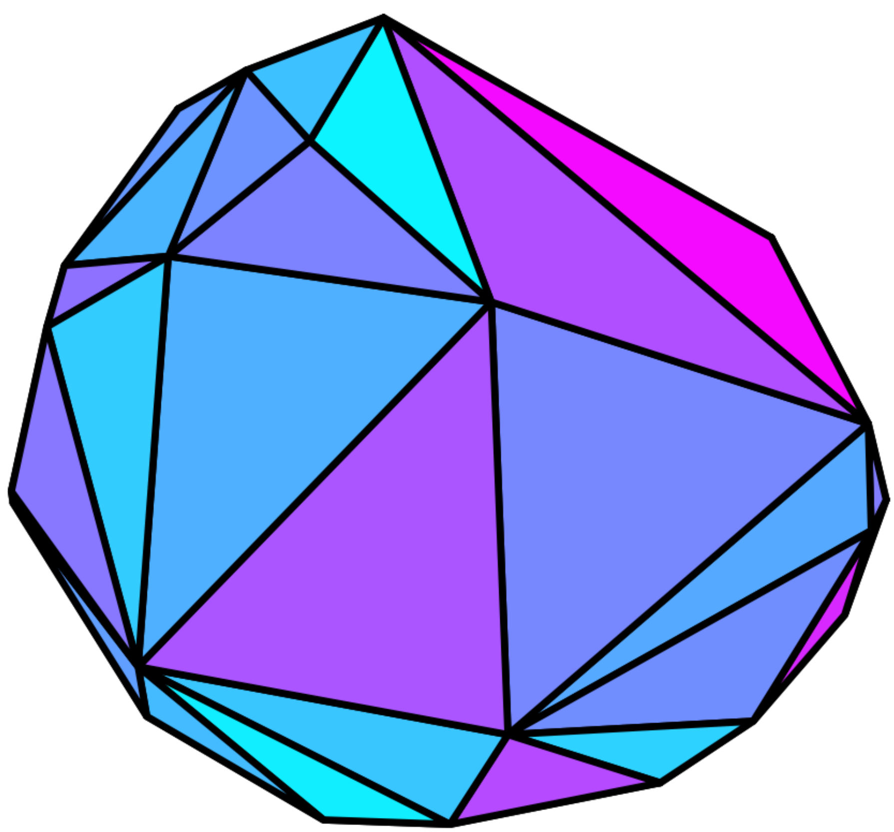
Given $a_1, \dots, a_m \in \mathbb{R}^d$ s.t. $0 \in CH(a_1, \dots, a_m)$

and $c \in \mathbb{R}^d$

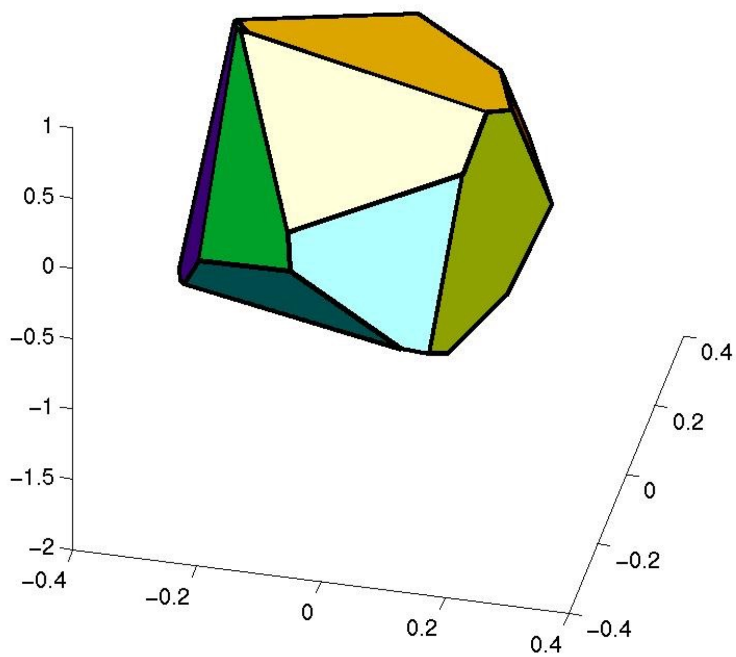
max α s.t. $\alpha c \in CH(a_1 \dots a_m)$

Here is a picture of the convex hull of random unit vectors in $\mathbb{R}^3$

If I choose Gaussian random vectors, some a probably inside the convex hull of the others



Random hyper-planes give polytopes like this:



To avoid strange behavior,

assume $c, a_1, \dots a_m$ are in <u>general position</u>

which we define to mean:

i. every subset of d is linearly independent, and

ii. every subset of $d+1$ is affinely independent.

(so no vector is in the affine span of d others)

This is true if add a slight random perturbation.

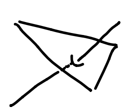
$a_1, \dots, a_d$ are linearly dependent iff $\sigma_d(a_1, \dots a_d) = 0$,

and this happens with probability 0 under Gaussian noise.

And, the probability one lies on the hyperplane defined by the affine span of d others $= 0$.

The definition of "General Position" depends on the context - so don't get too attached to one definition.

The consequence we want is that whenever the ray in direction c intersects a simplex $CH(a_i : i \in S)$ it intersects the relative interior of that simplex



<u>Carathéodory's Thm</u> If $y \in CH(a_1, \ldots, a_m)$

then $\exists\ |S| \leq d+1$ s.t. $y \in CH(a_i : i \in S)$

<u>proof.</u> Recall $y \in CH(a_1 \ldots a_m) \iff \exists\, t \in \mathbb{R}^m,\ 1^T t = 1,\ t \geq 0$

s.t. $y = \sum t(i) a_i$.

Assume, wolog, $t(i) > 0$ for all i and $m > d+1$

Will show can modify t to make an entry 0.

As $m > d+1$, the vectors $\begin{pmatrix} a_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} a_m \\ 1 \end{pmatrix}$ are

linearly dependant. So, $\exists\, r \in \mathbb{R}^m$, $r \neq \bar{0}$, s.t.

$\sum r(i) a_i = \bar{0}$ and $\sum r(i) = 0$.

So, for every μ, $\sum t(i) + \mu r(i) = 1$

and $\sum (t(i) + \mu r(i)) a_i = y$.

Will show can pick μ s.t. $t + \mu r \geq 0$ (1)

and $(t + \mu r)(i) = 0$ for some i. (2)

$\Rightarrow$ y is in CH of $m-1$ of the vectors.

(1) holds when $\mu = 0$.

As $r \neq \bar{0}$, is some very positive or very negative μ

for which (1) does not hold.

Choose μ to be largest (or smallest) μ for

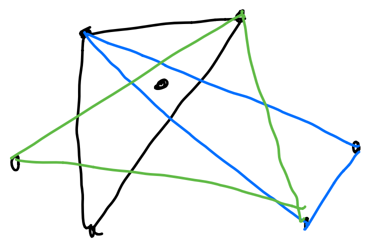
which (1) holds.

For this μ, $t + \mu r$ has a zero entry $\Rightarrow$ (2)

Note: this proof is algorithmic.

Given t we can efficiently find r and μ.

The set S need not be unique:



and are many more

The Dual LP:

Geometric view: if $\mathcal{H} = \{b : y^T b = 1\}$ is a hyperplane s.t. all of $a_1 \dots a_n$ is on one side, and $\alpha c \in \mathcal{H}$, then value of LP $\leq \alpha$.

$\alpha c \in \mathcal{H} \iff y^T(\alpha c) = 1 \implies \alpha = 1/y^T c$

So, LP is $\max y^T c$ s.t. $y^T a_i \leq 1, \ \forall i$

We just established

Weak Duality: value of dual upper bounds value of primal.

Algebraically: if $\alpha c = \sum t(i) a_i$, $\sum_i t(i) = 1$,

$$y^T(\alpha c) = \sum t(i)\, y^T a_i \leq \sum t(i) = 1$$

Strong Duality: optima of both are the same.

That is, if $\alpha_* = \max \alpha$ s.t. $\alpha c \in CH(a_1 \dots a_m)$

and y_* maximizes $y^T c$ s.t. $y^T a_i \le 1$

Then $y_*^T(\alpha c) = 1$.

<u>proof</u> Let the supporting hyperplane of $CH(a_1 \dots a_m)$ at $\alpha_* c$ be $\mathcal{H} = \{b : y^T b = 1\}$.

So, $y^T a_i \le 1$ for all i and $y^T(\alpha_* c) = 1$

The Simplex Method, or how to find max α s.t. $\alpha c \in CH(a_1, \dots a_m)$. Will involve solving many systems of equations. Will not have useful run-time guarantees.

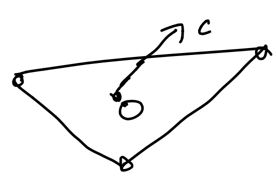
Start: Assume that $0 \in \text{interior}(CH(a_1, \dots a_m))$,

AND that we know t s.t. $\bar{0} = \sum_i t(i) a_i$

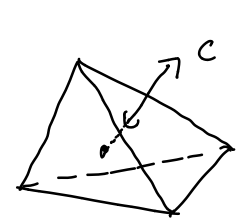
(We will later discuss how we find this t)

Find such a t that has $d+1$ nonzero coordinates.

So, $\bar{0} \in$ a simplex.



or



wolog, let $a_1, \dots, a_{d+1}$ be the corners

First step: find the point on the boundary hit by the ray in direction c. That is,

$$\max \alpha \quad \text{s.t.} \quad \alpha c \in CH(a_1, .. a_{d+1}).$$

There are only d+1 faces to check (each obtained by dropping a vertex).

If that face is $CH(a_1,..a_d)$,
then there is an α s.t. $\alpha c \in CH(a_1,..,a_d)$.

To find it, solve the equation

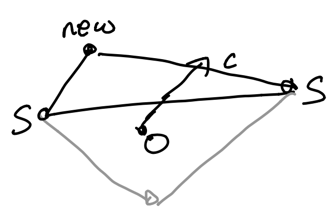
$$c = \sum_{i=1}^{d} x(i) a_i$$

It is the right face if all $x(i) \geq 0$

Let $\alpha = \left(\sum x(i)\right)^{-1}$, and $t(i) = \alpha x(i)$

So, $\alpha c = \sum_{i=1}^{d} t(i) a_i$.

General step: begin knowing $\alpha c = \sum_{i \in S} t(i) a_i$ for some $|S| = d$.



- a (d-1)-dim simplex.

Find an a_j s.t. $\hat{\alpha} c \in CH(a_j \cup (a_i)_{i \in S})$, for $\hat{\alpha} > \alpha$

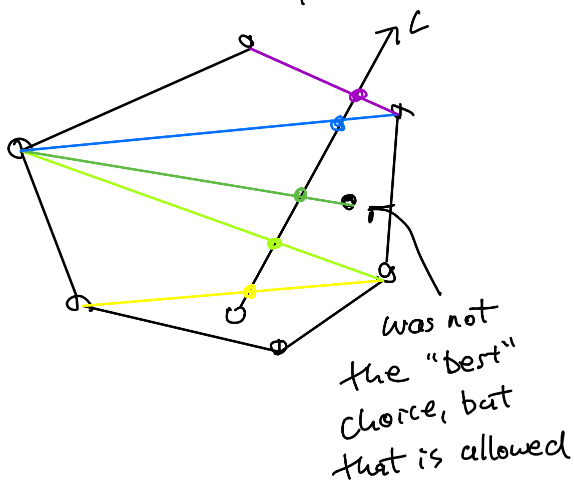
that is a simplex that contains the ray on the other side of the (d-1)-dim simplex.

We then proceed as before to find the point where they ray leaves this new simplex, and again express it as a positive sum of d of the points a_i.

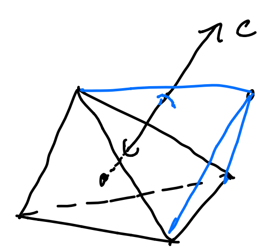
Note: our assumptions about "General Position" imply that αc will always be in the interior of the $(d-1)$ simplex, and thus $(\alpha+\varepsilon)c$ will be in the interior of the d-simplex, for small $\varepsilon > 0$.

We use the fact that a ray that goes through the interior of a simplex intersects its boundary at two points: where it enters and where it leaves.

A 2D example:



3D



When does it stop?
When there is no simplex on the other side.

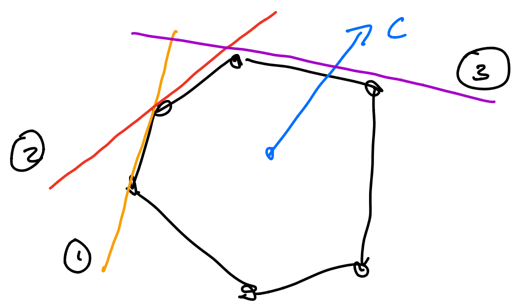
Let the current point be αc and $S \subseteq \{1..m\}$, $|S|=d$
index the simplex containing it.
$\alpha c \in$ relative interior $CH(a_i : i \in S)$
Claim: no simplex on other side $\Rightarrow$ αc on boundary of tope.
Let H = affine span $(a_i : i \in S)$
$= \{z : y^T z = 1\}$ for some y.
We have $y^T a_i = 1$ $i \in S$
and $y^T a_i \le 1$ for $i \notin S$.
So, H is a supporting hyperplane of $CH(a_1, \dots a_m)$ at αc,
$\Rightarrow$ α is the optimum

The Dual simplex method:
Find y maximizing $y^T c$ s.t. $y^T a_i \le 1$ $\forall i$.
This is the dual LP: $\max y^T c$
$y^T a_i \le 1$, $\forall i$

Imagine that know some supporting
hyperplane: y s.t. $y^T a_i = 1$ $i \in S$, $|S|=d$
and $y^T a_i \le 1$ $\forall i$

Want to rotate it around the polytope to find a supporting hyperplane that increases $y^T c$.

In 2D, looks like this



To see is alway possible, if y is not optimal, look at intersection of αc in that plane, and the simplex: