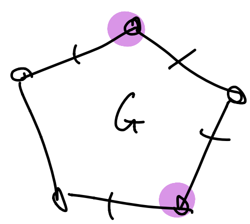
Max Cut: We will use convex programming (linear programming over semidefinite matrices) to get a 0.878... approximation of maxcut. This is a famous result of Goemans & Williamson

Input: graph $G=(V,E)$.
For $S \subset V$, define $\mathrm{cut}(S) = \#\{(a,b) \text{ s.t. } |\{a,b\} \cap S| = 1\}$

$$\mathrm{maxcut}(G) = \max_S \mathrm{cut}(S).$$



$\mathrm{cut}(S) = 4$

In our previous notation, $\mathrm{cut}(S) = |\partial(S)|$

Easy results first. Define $m = |E|$.

Lem 1 $\mathrm{maxcut}(G) \geq m/2$

proof Consider choosing S unif. at random
$\forall a \in V$ $\Pr[a \in S] = \frac{1}{2}$, independently.
For $\forall (a,b) \in E$

$$\begin{aligned}\Pr[(a,b) \in \partial(S)] &= \Pr[(a \in S \text{ and } b \notin S) \text{ or } (a \notin S \text{ and } b \in S)] \\ &= \Pr[a \in S \text{ and } b \notin S] + \Pr[a \notin S \text{ and } b \in S] \\ &= 1/4 + 1/4 = 1/2\end{aligned}$$

$$\mathop{\mathbb{E}}_S \mathrm{cut}(S) = \sum_{(a,b) \in E} \Pr[(a,b) \in \partial(S)] = \frac{m}{2}$$

max $\geq$ average, so $\exists S$ s.t. $\mathrm{cut}(S) \geq m/2$

Can turn this into an algorithm. But there's a simpler algorithm.

Let's describe it in terms of $\{\pm 1\}^n$ vectors, where $n = |V|$.

$x \in \{\pm 1\}^n$, $S_x = \{a : x(a) = 1\}$

<u>Local Search</u>

Start with any $x \in \{\pm 1\}^n$ (like $\mathbb{1}$, or random)

while $\exists a$ s.t. $x(a) \sum_{b:(a,b) \in E} x(b) > 0$ — most neighbors of a are on same side

$x(a) = -x(a)$.

Return x (or $S_x = \{a : x(a) = 1\}$)

Idea: if moving a into or out of S increases the cut, do it.

<u>Claim 1</u> $\mathrm{cut}(S_x) = \frac{1}{2} \sum_{(a,b) \in E} 1 - x(a)x(b)$

<u>proof</u>

<u>Claim 2</u> If $\hat{x}$ is vector after moving a,

$$\mathrm{cut}(S_{\hat{x}}) = \mathrm{cut}(S_x) + x(a) \sum_{b:(a,b) \in E} x(b)$$

Can turn this into an algorithm. But there's a simpler algorithm.

Let's describe it in terms of $\{\pm 1\}^n$ vectors, where $n = |U|$.

<u>Local Search</u>

Start with any $x \in \{\pm 1\}^n$ (like $\mathbb{1}$, or random)

while $\exists a$ s.t. $x(a) \sum_{b : (a,b) \in E} x(b) > 0$

$x(a) = -x(a)$.

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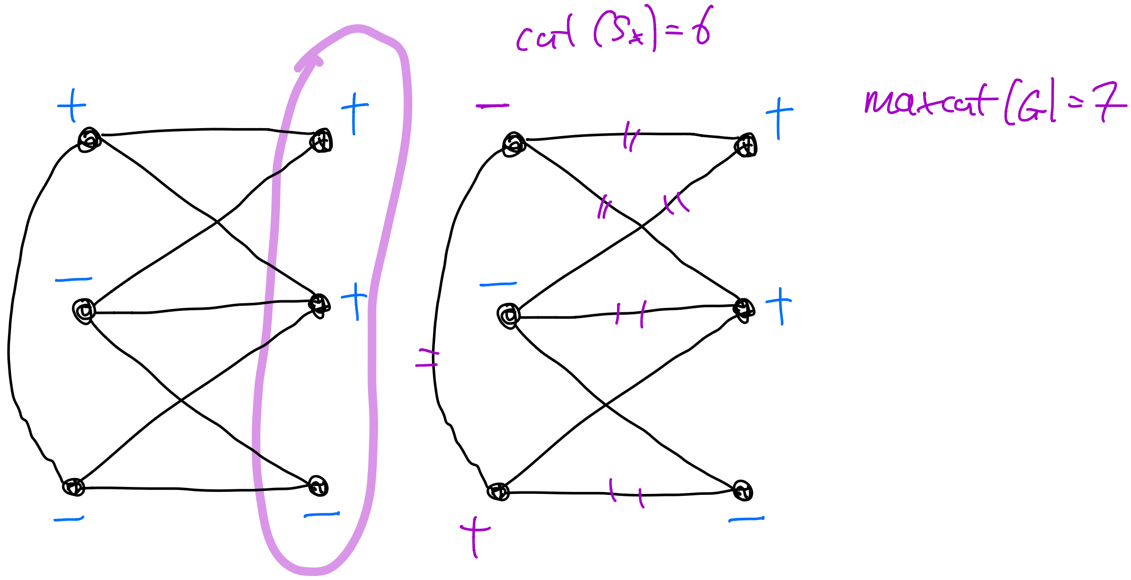

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Local Search

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Idea: if moving a into or out of S increases the cut, do it.

Claim 1 $\quad \mathrm{cut}(S_x) = \frac{1}{2} \sum_{(a,b)\in E} 1 - x(a)x(b)$ $\qquad \frac{1}{2}(1-(-1)) = 1$

proof $\quad x(a)x(b) = -1$ if $(a,b) \in \partial(S_x)$

$= 1$ o.w. $\qquad \frac{1}{2}(1-1) = 0$

Claim 2 $\quad$ If $\hat{x}$ is vector after moving a,

$\mathrm{cut}(S_{\hat{x}}) = \mathrm{cut}(S_x) + x(a) \sum_{b:(a,b)\in E} x(b)$

x: $\oplus$ —— $\oplus$ $\qquad$ $\tilde{x}$: $\ominus$ —— $\oplus$

$\text{cut}(S_x) = 0$ $\qquad$ $\text{cut}(S_{\tilde{x}}) = 1$

$x(a)x(b) = 1$

Lem 2 Local Search terminates, and returns x with $\text{cut}(S_x) \geq \frac{m}{2}$

proof

When stops $\forall a$ $\sum_{b:(a,b)\in E} x(a)x(b) \leq 0$

$$\text{cut}(S_x) = \frac{1}{2}\sum_{(a,b)\in E} 1 - x(a)x(b) = \frac{m}{2} - \frac{1}{2}\sum_{(a,b)\in E} x(a)x(b) \quad \text{cuts} \geq \frac{m}{2}$$

need: $\sum_{(a,b)\in E} x(a)x(b) \leq 0$

$$\sum_{(a,b)\in E} x(a)x(b) = \frac{1}{2}\sum_{a}\sum_{b:(a,b)\in E} x(a)x(b) \leq 0$$

How to do better?

Goemans & Williamson '95: relax $x(a) \in \{\pm 1\}$

replace with $v_a \in \mathbb{R}^n$, $\|v_a\|_2 = 1$

$$x(a)x(b) \rightarrow v_a^T v_b$$

$$\sum_{(a,b)\in E} 1 - x(a)x(b) \rightarrow \sum_{(a,b)\in E} 1 - v_a^T v_b$$

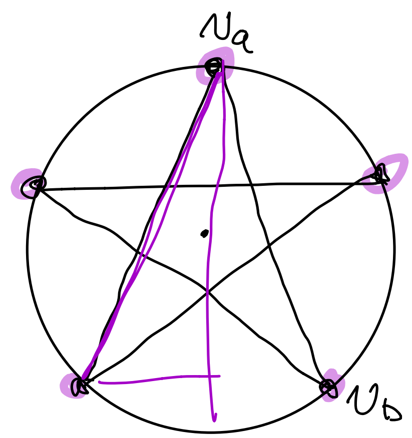
$v_a = 0$

$\frac{1}{2} m$

Solve the vector problem

$$UP(G) = \max \frac{1}{2} \sum_{(a,b)\in E} 1 - v_a^T v_b \quad \text{s.t.} \quad \|v_a\|_2 = 1$$

Ex.



$(a,b) \in E$

$v_a^T v_b = \cos\left(\frac{4\pi}{5}\right) \approx -0.81$

for all $(a,b) \in E$

$$\frac{1}{2} \sum_{(a,b)\in E} 1 - v_a^T v_b \approx 4.52 > \text{maxcut}(G) = 4$$

1. We can turn the solution into an approximate solution to maxcut.
2. We can approximately solve UP in polynomial time.

Claim 3 $VP(G) \ge \text{maxcut}(G)$ $x(a) \in \pm 1, \forall a$

proof consider $v_a = u \cdot x(a)$ for any unit vector u.

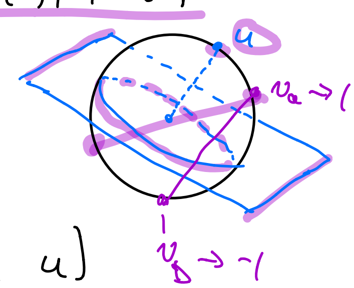
Now, $v_a^T v_b = x(a)x(b)$.

But we can choose v_a differently, and get a larger value. $\xleftarrow{-u} \quad \xrightarrow{u}$

To round vectors $v_1, \dots, v_n$ into ± 1 $x(1), \dots, x(n)$:

Choose a random unit vector u,

set $x(a) = \begin{cases} 1 & \text{if } u^T v_a \ge 0 \\ -1 & \text{o.w.} \end{cases}$

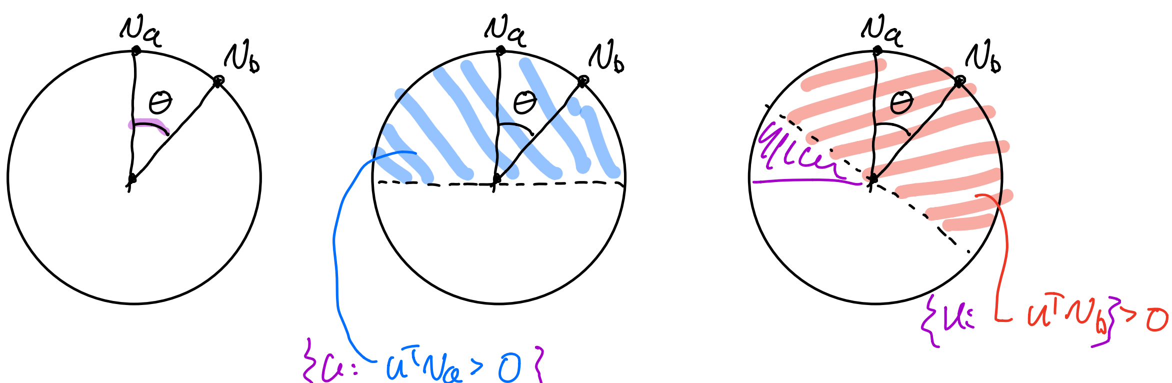


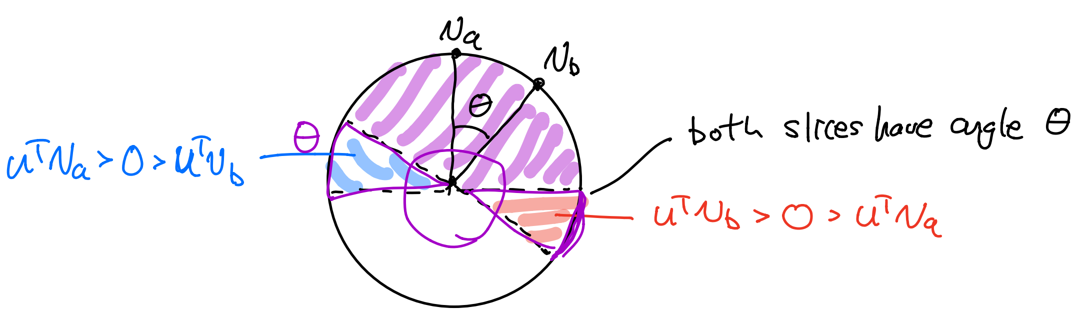
(is equivalent to use Gaussian random u)

$r \sim N(0, I_n)$ $u = r / \|r\|$

Claim 4 $\Pr_r[(a,b) \in \partial(S_x)] = \frac{1}{\pi}\text{ang}(v_a, v_b) = \frac{\theta}{\pi} = \frac{2\theta}{2\pi}$

proof First, look at the 2D case





In general, project u to span(v_a, v_b), and apply this analysis.

S_u or S_{x_u} $\qquad v_a^T v_b = \cos(\theta)$

Claim 5 $\mathbb{E}_u \operatorname{cut}(S_x) = \frac{1}{\pi} \sum_{(a,b)\in E} \operatorname{acos}(v_a^T v_b)$

proof $v_a^T v_b = \cos\theta$, $\theta = \operatorname{acos}(v_a^T v_b)$

Claim 6 $\min_{-1 \le t \le 1} \dfrac{\frac{1}{\pi}\operatorname{acos}(t)}{\frac{1}{2}(1-t)} \ge 0.878$

Theorem $\mathbb{E}_u \operatorname{cut}(S_x) \ge 0.878 \cdot \operatorname{maxcut}(G)$

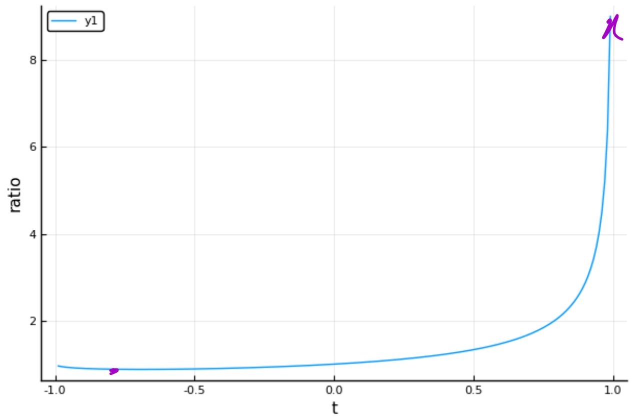
proof
$$\begin{aligned}\mathbb{E}_u \operatorname{cut}(S_x) &= \frac{1}{\pi}\sum_{(a,b)\in E} \operatorname{acos}(v_a^T v_b) \\ &\ge 0.878 \sum_{(a,b)\in E} \frac{1}{2}(1 - v_a^T v_b) \\ &= 0.878\ \mathrm{UP}(G) \\ &\ge 0.878\ \operatorname{maxcut}(G)\end{aligned}$$

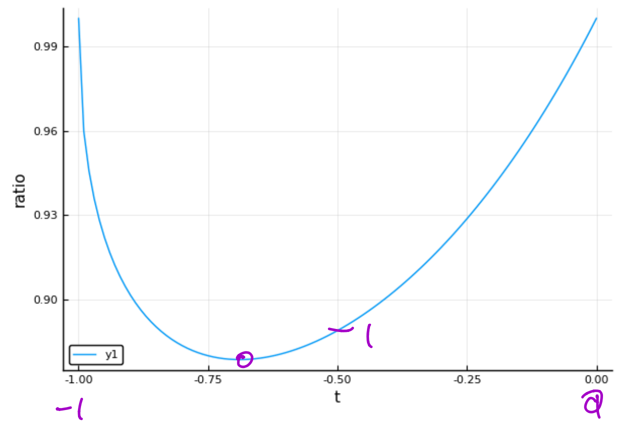
Proof of claim 6

Change to $\theta = \cos(t)$, compute deriv, set to 0

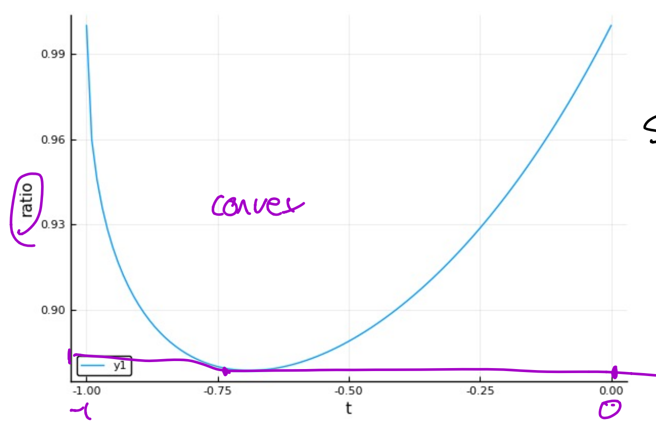
crit point $\cos(\theta) + \theta\sin(\theta) = 1$

or, 2. plot it.

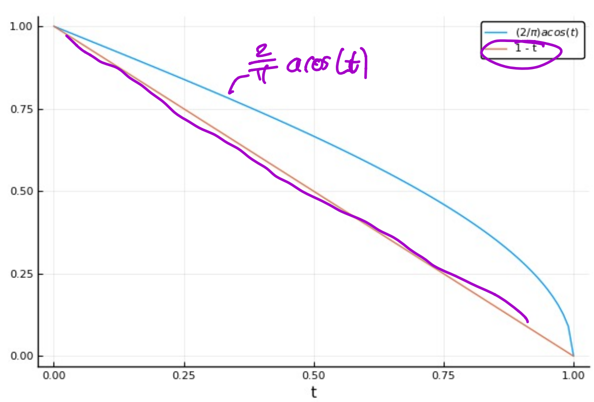




3. Use the plot to derive the bound. (see GW)



here, ratio is convex. So use a supporting plane to get a lower bound



here, ratio ≥ 1

How to solve VP:

$$\max_{v_1,\dots,v_n} \frac{1}{2} \sum_{(a,b)} 1 - v_a^T v_b \quad \text{s.t. } v_a^T v_a = 1, \text{ for all } a.$$

This problem is linear in the Gram matrix:

$$M(a,b) = v_a^T v_b \qquad M = V^T V \quad \text{where } V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$

problem becomes $\max \frac{1}{2} \sum_{(a,b)\in E} 1 - M(a,b)$ s.t. $M(a,a) = 1$, $\forall a$

and M is a Gram matrix.

Claim:

M is a Gram matrix iff M is positive semidefinite.

proof $M = U^T U \Rightarrow x^T M x = (Ux)^T (Ux) \geq 0$, $\forall x$

If M is psd, can find a "Cholesky" - factorization

$M = LL^T \qquad L = \begin{pmatrix} \ddots & 0 \\ * & \ddots \end{pmatrix} \quad \left(\begin{array}{c|c} \ddots \; * & 0 \end{array}\right)$

$V = L^T$ M is Gram matrix of L^T

eigen: $M = U \Lambda U^T$ $\Lambda = \text{diag, non-neg}$ $V = \Lambda^{1/2} U^T$ $U^T U = M$

So, solve

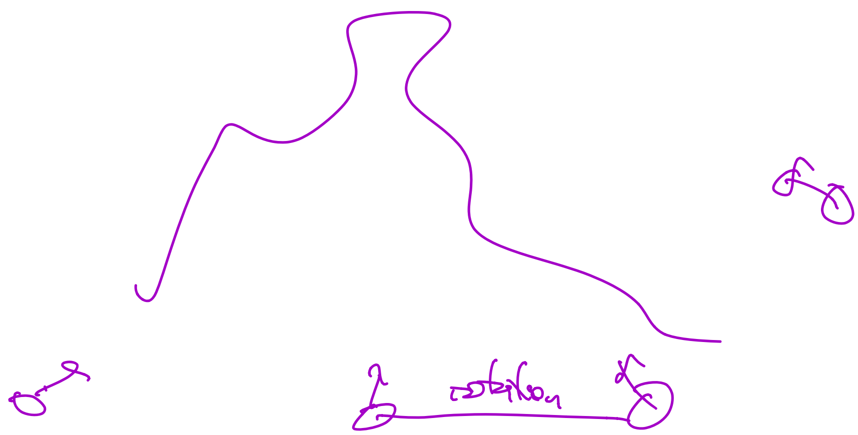
$$\max \frac{1}{2} \sum_{(a,b)\in E} 1 - M(a,b) \quad \text{s.t. } M \succeq 0, \; M(a,a) = 1, \; \forall a$$

Cholesky factor $M = LL^T$ eigen decomp

$x = \text{sign}(u^T V)$ for a random vector u

Can find graphs for which alg
does not beat 0.879

UGC $\Longrightarrow$ can't beat GW.



variables are $\hat{M}^n$ n^2 variables...
for many problems solution has low rank
are many special purpose algs.

Semidefinite programming

$$\max \operatorname{Tr}(C^T M) = \langle C, M \rangle$$
$$\text{s.t. } M \succeq 0 \quad \text{pos def}$$
$$\operatorname{Tr}(B_i^T \underline{M}) \le b_i$$

$N(0,1)$ $\quad p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$x = x_1, \ldots, x_n$

$$P(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2}$$

$$= (2\pi)^{-n/2} \prod_i e^{-x_i^2/2}$$

$$= (2\pi)^{-n/2} e^{-\sum x_i^2/2}$$

$$= \text{"} \quad e^{-\frac{1}{2}\|x\|^2}$$