Max Cut: We will use convex programming (linear programming over semidefinite matrices) to get a 0.878-approximation of maxcut. This is a famous result of Goemans & Williamson.

Input: graph $G = (V,E)$. For $S \subseteq V$, define $\text{cut}(S) = \sum_{(i,j) \in E, i \in S, j \notin S} 1$. 

\[
\max_{S} \text{cut}(S) = \frac{m}{2}
\]

In our previous notation, $\text{cut}(S) = |\partial(S)|$.

Easy results first. Define $m = |E|$.

**Lemma** $\max_{S} \text{cut}(S) \geq \frac{m}{2}$

**Proof** Consider choosing $S$ uniformly at random. Then $\text{Pr}[i \in S] = \frac{1}{2}$, independently. For $(a,b) \in E$,

\[
\text{Pr}[(a,b) \in \partial(S)] = \text{Pr}[[a \in S \text{ and } b \notin S] \text{ or } [a \notin S \text{ and } b \in S]]
\]

\[
= \text{Pr}[[a \in S \text{ and } b \notin S]] + \text{Pr}[[a \notin S \text{ and } b \in S]]
\]

\[
= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}
\]

\[
\sum_{S} \text{Pr}[(a,b) \in \partial(S)] = \frac{m}{2}
\]

max Satellite, so There $S$ s.t. $\text{cut}(S) \geq \frac{m}{2}$
Can turn this into an algorithm. But there's a simpler algorithm.

Let's describe it in terms of $\{\pm 1\}^n$ vectors, where $n=|V|$. 
\[ x \in \{\pm 1\}^n, \quad S_x = \{a : x(a) = 1\} \]

**Local Search**

Start with any $x \in \{\pm 1\}^n$ [like 1, or random]

while $\exists a$ s.t. $x(a) \sum_{b : (a,b) \in E} x(b) > 0$  
\[ x(a) = -x(a). \]

Return $x$ (or $S_x = \{a : x(a) = 1\}$)

Idea: if moving $a$ into or out of $S$ increases the cut, do it.

**Claim 1**  
\[ \text{cut}(S_x) = \frac{1}{2} \sum_{(a,b) \in E} (1 - x(a) x(b)) \]

**proof**

**Claim 2** If $x$ is vector after moving $a$, 
\[ \text{cut}(S_x) = \text{cut}(S_x) + x(a) \sum_{b : (a,b) \in E} x(b) \]
Can turn this into an algorithm. But there’s a simpler algorithm.

Let’s describe it in terms of \(\{\pm 1\}^n\) vectors, where \(n = |V|\).

**Local Search**

Start with any \(x \in \{\pm 1\}^n\) (like \(1^n\) or random)

while \(\exists a \text{ s.t. } x(a) \sum_{b: a \in b} x(b) > 0\)

\[x(a) = -x(a).\]

Return \(x\) (or \(S_x = \{a : x(a) = 1\}\))

Idea: if moving \(a\) into or out of \(S\) increases the cut, do it.

\[\text{cut}(S_x) = 6\]

\[\text{maxcut}(G) = 7\]
Can turn this into an algorithm. But there's a simpler algorithm.

Let's describe it in terms of \( \pm 1 \) vectors, where \( n = |U| \).

**Local Search**

Start with any \( x \in \pm 1^n \) (like \( 1 \), or random)

while \( \exists a \text{ s.t. } x(a) \Sigma x(b) > 0 \) \( \forall b : (a,b) \in E \)

\[ x(a) = - x(a) \]

Return \( x \) (or \( S_x = \{ \alpha : x(\alpha) = 1 \} \))

Idea: if moving \( a \) into or out of \( S \) increases the cat, do it.

**Claim 1** \( \text{cut}(S_x) = \frac{1}{2} \sum_{(a,b) \in E} 1 - x(a)x(b) \) \( \frac{1}{2}(1-(-1)^{|E|}) \)

**Proof** \( x(a)x(b) = -1 \) if \( (a,b) \in \partial(S_x) \)

= 1 o.w. \( \frac{1}{2}(1-1) = 0 \)

**Claim 2** If \( \hat{x} \) is vector after moving \( a \),

\[ \text{cut}(S_{\hat{x}}) = \text{cut}(S_x) + x(a) \sum_{b : (a,b) \in E} x(b) \]

Proof: \( x(a)x(b) = -1 \) if \( (a,b) \in \partial(S_x) \)

= 1 o.w. \( \frac{1}{2}(1-1) = 0 \)
\[ x: \text{catch} \quad \text{cut} \quad \text{CSI} \quad x: \text{catch} \]

\[ \text{cat}(S_x) = 0 \quad \text{cat}(S_x) = 1 \]

\[ x(a) \cdot x(b) = 1 \]

\[ \text{lem 2} \quad \text{local search terminates, and returns} \]

\[ x \quad \text{with} \quad \text{cat}(S_x) \geq \frac{m}{2} \]

\[ \text{proof} \]

When stops \( t \in \{a, b\} \sum_{b: (a, b) \in E} x(a) \cdot x(b) \leq 0 \)

\[ \text{cat}(S_x) = \frac{1}{2} \sum_{(a, b) \in E} 1 - x(a) \cdot x(b) = \frac{m}{2} - \frac{1}{2} \sum_{(a, b) \in E} x(a) \cdot x(b) \quad x \in \{a, b\} \]

\[ \text{need:} \quad \sum_{(a, b) \in E} x(a) \cdot x(b) \leq 0 \]

\[ \sum_{(a, b) \in E} x(a) \cdot x(b) = \frac{1}{2} \sum_{a} \sum_{b: (a, b) \in E} x(a) \cdot x(b) \leq 0 \]
How to do better?

Goemans & Williamson 95: relax $x(a) \in \{\pm 1\}$ replace with $u_a \in \mathbb{R}^n$, $\|u\|_2 = 1$

\[
x(a)x(b) \rightarrow u_a^T u_b
\]

\[
\sum_{(a,b) \in E} (-x(a)x(b)) \rightarrow \sum_{(a,b) \in E} 1 - u_a^T u_b
\]

Solve the vector problem

\[
\text{UP}(\mathcal{G}) = \max \frac{1}{2} \sum_{(a,b) \in E} 1 - u_a^T u_b \quad \text{s.t.} \quad \|u\|_2 = 1
\]

Ex.

\[
u_a = 0
\]

\[
\frac{1}{2} u
\]

\[
u_a^T u_b = \cos \left( \frac{4\pi}{5} \right) \approx -0.81
\]

for all $(a,b) \in E$

\[
\frac{1}{2} \sum_{(a,b) \in E} 1 - u_a^T u_b \approx 4.52 > \text{maxcut} (\mathcal{G}) = 4
\]

1. We can turn the solution into an approximate solution to maxcut.

2. We can approximately solve UP in polynomial time.
Claim 3 \( VPL(G) \geq \maxcut(G) \)

\[ x(e) \in \{ \pm 1 \} \]

**Proof:** consider \( x_a = u \cdot x(a) \) for any unit vector \( u \).

Now, \( x_a \cdot x_b = x(a) \cdot x(b) \).

But we can choose \( x_a \) differently, and get

\[ e \cdot u - u \]

To round vectors \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) into \( \pm x(1), \ldots, x(n) \):

choose a random unit vector \( u \),

set \( x(a) = \begin{cases} 1 & \text{if } u^\top \mathbf{v}_a > 0 \\ -1 & \text{o.w.} \end{cases} \)

(is equivalent to use Gaussian random \( u \))

\( u \sim N(0, I) \), \( u = \frac{r}{\|r\|} \)

Claim 4 \( \Pr\left[ (a,b) \in \mathcal{E}(S x) \right] = \frac{1}{\pi} \| \mathbf{v}_a \cdot \mathbf{v}_b \| = \frac{\Theta}{\pi} = \frac{2\Theta}{2\pi} \)

**Proof:** First, look at the 2D case.
In general, project \( u \) to span \((v_a, v_b)\), and apply this analysis.

\[
\begin{align*}
\langle s_u \rangle & \quad \text{or} \quad \langle s_{x_u} \rangle \\
\theta & = 0.878
\end{align*}
\]

**Claim 5** \( \text{Cut}(S_x) = \frac{1}{\mathbb{E}} \sum \cos(v_a^T v_b) \)

**Proof** \( v_a^T v_b = \cos \theta \), \( \theta = \cos(v_a^T v_b) \)

**Claim 6** \( \min_{-1 \leq t \leq 1} \frac{1}{2} \mathbb{E} \cos(t) \geq 0.878 \)

**Theorem** \( \mathbb{E} \text{Cut}(S_x) \geq 0.878 \cdot \text{maxCut}(G) \)

**Proof** \( \mathbb{E} \text{Cut}(S_x) = \frac{1}{\mathbb{E}} \sum \cos(v_a^T v_b) \)

\[
\geq 0.878 \sum_{(i,j) \in E} \frac{1}{2} (1 - v_a^T v_b) \\
= 0.878 \ \text{UP}(G) \\
\geq 0.878 \ \text{maxCut}(G)
\]

**Proof of Claim 6**

- Change to \( \theta = \cos(t) \), compute derivative, set to 0
- Critical point \( \cos(\theta) + \theta \sin(\theta) = 1 \)
or 2. plot it.

3. Use the plot to derive the bound. (see Gu)

here, ratio is convex. So use a supporting plane to get a lower bound

here, ratio ≥ 1
How to solve UP:

$$\max_{u_1, \ldots, u_n} \frac{1}{2} \sum_{(i,j)} 1 - u_i^T u_j \quad \text{s.t.} \quad u_i^T u_i = 1, \quad \text{for all } i.$$

This problem is linear in the Gram matrix:

$$M(a,b) = u_a^T u_b \quad \text{where } U = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}$$

problem becomes

$$\max \frac{1}{2} \sum_{(i,j)\in E} 1 - M(a,b) \quad \text{s.t. } M(a,a) = 1, \forall a$$

and $M$ is a Gram matrix.

Claim:

$M$ is a Gram matrix iff $M$ is positive semidefinite.

Proof:

$M = U^T U \Rightarrow \forall Mx = (U x)^T (U x) \geq 0, \forall x$

If $M$ is PSD, can find a "Cholesky" factorization

$M = L L^T \quad \text{where } L = \begin{pmatrix} \ast & 0 \\ \ast & \ast\end{pmatrix} \begin{pmatrix} \ast & 0 \\ \ast & \ast\end{pmatrix}$

$V = L^T \quad \text{is Gram matrix of } L$

Eigens:

$M = U \Lambda U^T \quad \Lambda = \text{diag(1, \text{non-neg})} \quad V = \sqrt{\Lambda} U \quad U^T U = M$

So, solve

$$\max \frac{1}{2} \sum_{(a,b)\in E} 1 - M(a,b) \quad \text{s.t. } M \geq 0, \quad M(a,a) = 1, \forall a$$

Cholesky factor $M = L L^T$ even decomposable $X = \text{sign}(u^T V)$ for a random vector $u$
can find graphs for which alg does not beat 0.879

UGC \implies \text{can't beat EW.}

variables are $M^n$ n^2 variables...
for many problems solution has low rank
are many special purpose algs.

Semidefinite programming

$$\max \quad \text{Tr}(C^TM) = \langle C, M \rangle$$
$$\text{subject to: } M \succeq 0 \quad \text{pos def}$$
$$\text{Tr} \left( B_i M \right) \leq b_i$$
\[ N(0, I) \quad p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \]

\[ x = x_1, \ldots, x_n \]

\[ p(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} \]

\[ = (2\pi)^{-n/2} \prod_{i} e^{-x_i^2/2} \]

\[ = (2\pi)^{-n/2} e^{-\frac{1}{2} \|x\|^2} \]

\[ = " \quad e^{-\frac{1}{2} \|x\|^2} \]