Sparse recovery 2: Recovering sparse signals with noise.

Problem: Given d-by-n A, $d \ll n$ $\quad d\left(\overset{n}{A}\right)$
and a vector $b \in \mathbb{R}^d$, try to find a
sparse x s.t. $Ax \approx b$.

Formally:

x_0

Given $\varepsilon > 0$, $\min \|x\|_0$ s.t. $\|Ax - b\|_2 \le \varepsilon$ $\quad (P_0)$

Is NP-hard, but can approximately solve it for nice A.

convex

Let x_* be the solution to $\quad$ conve

$\min \|x\|_1$ s.t. $\|Ax - b\|_2 \le \varepsilon$. $\quad (P_1)$

We will show x_* is close to the solution to P_0.

$d = \text{const} \cdot n$ $\qquad$ as $\frac{s}{d} \to 0$, $\frac{d}{n}$ fixed
$s = \text{const} \cdot d$ $\qquad$ $\delta \to 0$

Nice: A satisfies the (s,δ)-Restricted Isometry Property
if for all x s.t. $\|x\|_0 \le s$ $\quad$ usually are squares
$(1-\delta)\|x\|_2 \le \|Ax\|_2 \le (1+\delta)\|x\|_2$

Def $\delta_s(A)$ = smallest δ s.t. A satisfies (s,δ)-RIP

Theorem If $\delta_{3s} + \frac{1}{\sqrt{2}}\delta_{2s} \le 1 - 1/\sqrt{2}$, $b = Ax_0 + e$

with $\|x_0\|_0 \le s$ and $\|e\|_2 \le \varepsilon$,

then x_*, the solution to P_1, satisfies

$$\|x_0 - x_*\|_2 \le \frac{2\sqrt{2}\,\varepsilon}{1 - 1/\sqrt{2} - \delta_{3s} - \frac{1}{\sqrt{2}}\delta_{2s}} \quad \text{const}$$

Notation: let $S = \mathrm{supp}(x_0)$, $s = |S|$, $T = \{1,\dots,n\} - S$.

$h = x_* - x_0$ $\quad \|h\|_2 < ?$ $\quad x_0(i) = 0 \quad i \in T$

Claim 1 $\|Ah\|_2 \le 2\varepsilon$ $\quad$ USE RIP $\quad \|h\|_2 \le ?\ \|Ah\|_2$

proof $Ah = Ax_* - Ax_0 = (Ax_* - b) - (Ax_0 - b)$

so $\|Ah\|_2 \le \underbrace{\|Ax_* - b\|_2}_{< \varepsilon} + \underbrace{\|Ax_0 - b\|_2}_{< \varepsilon} \le 2\varepsilon$

Let $h(S)$ be the restriction of h to coordinates in S.

(or just zero out coordinates not in S)

$T = \{1..n\} - S$

Claim 2 $\|h(S)\|_1 \ge \|h(T)\|_1$

proof $\|x_0\|_1 \ge \|x_*\|_1$ because x_* min $\|x\|_1$ s.t. ...

$= \|x_0 + h\|_1 = \|(x_0+h)(S)\|_1 + \|(x_0+h)(T)\|_1$

Δ-ineq $\qquad x_0(T) = 0$

$\ge \|x_0(S)\|_1 - \|h(S)\|_1 + \|h(T)\|_1$

$= \|x_0\|_1 - \|h(S)\|_1 + \|h(T)\|_1$

$\Rightarrow \|h(S)\|_1 \ge \|h(T)\|_1$

Would also like to say $\|h(S)\|_2 \geq \text{const} \cdot \|h(T)\|_2$,

but might not be true.

Ex. $h(S) = \underbrace{\frac{1}{s}, \dots, \frac{1}{s}}$, s coords $\|h(S)\|_1 = 1$, $\|h(S)\|_2 = \frac{1}{\sqrt{s}}$

$h(T) = 1, 0, 0, \dots, 0$ $\|h(T)\|_1 = 1$, $\|h(T)\|_2 = 1$

Will show something like this after excluding $2s$ coordinates from T.

Def. T_1 = the $2s$ coords in T on which h is largest

$R = T - T_1 = \{1, \dots, n\} - S - T_1$

Claim 3 $\|h(R)\|_2^2 \leq \frac{1}{2} \|h(S)\|_2^2$ (only uses claim 2)

proof $\|v\|_2^2 \leq \|v\|_\infty \|v\|_1$

$\left(\sum_i v(i)^2 \leq \sum_i |v(i)| \cdot \|v\|_\infty = \|v\|_1 \cdot \|v\|_\infty \right.$

$\|h(R)\|_\infty \leq \min h(T_1) \leq \frac{1}{2s} \|h(T_1)\|_1$ (min $\leq$ average)

$\leq \frac{1}{2s} \|h(T)\|_1$ $T_1 \subset T$

$\|h(R)\|_1 \leq \|h(T)\|_1$ $R \subset T$

$\|h(R)\|_2^2 \leq \frac{1}{2s} \|h(T)\|_1^2 \leq \frac{1}{2s} \|h(S)\|_1^2$

$\|v\|_1 \leq \sqrt{\|v\|_0} \|v\|_2, \forall v$

$\leq \frac{s}{2s} \|h(S)\|_2^2$

$= \frac{1}{2} \|h(S)\|_2^2$

We also want to show $\|A h(R)\|_2 \leq$ something.

Will need RIP, but only applies to sparse vectors.

$T_1 = 2s$ coords of T on which h is largest

Let T_2 = largest $2s$ coords of h in $T - T_1$

T_3 = " " " " " " $T - T_1 - T_2$

etc.

So, $T = T_1 \cup T_2 \cup \cdots \cup T_k$, some k, and T_k can be smaller.

$R = T_2 \cup \cdots \cup T_k$ $\{1, \dots, n\} = S \cup T_1 \cup R$

__Claim 4__ $\sum_{i \geq 2} \|h(T_i)\|_2 \leq 1/\sqrt{2}\, \|h(S)\|_2$ (wts $\|h\|_2 \leq$?)

__proof__ For $i \geq 2$, $\|h(T_i)\|_2^2 \leq \|h(T_i)\|_\infty \|h(T_i)\|_1$

$\leq \|h(T_i)\|_\infty \|h(T_{i-1})\|_1$ h decreases as $i \uparrow$

$\leq \frac{1}{2s} \|h(T_{i-1})\|_1^2$ $\|h(T_i)\|_\infty \leq \min \underline{h(T_{i-1})}$ $\frac{1}{2s}\|h(T_{i-1})\|$

$\Rightarrow \sum_{i=2}^{k} \|h(T_i)\|_2 \leq 1/\sqrt{2s} \sum_{i=1}^{k-1} \|h(T_i)\|_1$

$= 1/\sqrt{2s}\, \|h(T_1 \cup \cdots \cup T_{k-1})\|_1 \leq 1/\sqrt{2s}\, \|h(T)\|_1$

$\leq 1/\sqrt{2s}\, \|h(S)\|_1$ (claim 2)

$\leq 1/\sqrt{2}\, \|h(S)\|_2$, as $\|h(S)\|_0 \leq s$

__Claim 5__ $\|A\, h(R)\|_2 \leq \frac{1}{\sqrt{2}} (1 + \delta_{2s})\, \|h(S)\|_2$

$\|A\, h(R)\|_2 \underset{\Delta}{\leq} \sum_{i=2}^{k} \|A\, h(T_i)\|_2 \leq (1 + \delta_{2s}) \sum_{i \geq 2}^{k} \|h(T_i)\|_2$

$\leq (1 + \delta_{2s}) \frac{1}{\sqrt{2}} \|h(S)\|_2$

proof of theorem $\{1,\dots,n\} = S \cup T_1 \cup R$

$$\|h\|_2 = \|x_0 - x_*\|_2 \le \frac{2\sqrt{2}\,\varepsilon}{1 - 1/\sqrt{2} - \delta_{3s} - \frac{1}{\sqrt{2}}\delta_{2s}}$$

$$\|Ah\|_2 \le 2\varepsilon$$

$$\|A\,h(R)\|_2 \le \frac{1}{\sqrt{2}}(1+\delta_{2s})\|h(S)\|_2 \le \frac{1}{\sqrt{2}}(1+\delta_{2s})\|h(S\cup T_1)\|_2$$

Δ-ineq $\quad 2\varepsilon \ge \|Ah\|_2 \ge \|A\,h(S\cup T_1)\|_2 - \|Ah(R)\|_2$

$|S\cup T_1| = 3s$

$$\ge (1-\delta_{3s})\|h(S\cup T_1)\|_2 - \frac{1}{\sqrt{2}}(1+\delta_{2s})\|h(S\cup T_1)\|_2$$

$$= \left[1 - 1/\sqrt{2} - \delta_{3s} - \frac{1}{\sqrt{2}}\delta_{2s}\right]\|h(S\cup T_1)\|_2$$

$$\|h(S\cup T_1)\|_2^2 = \|h\|_2^2 - \|h(R)\|_2^2$$
$$\ge \|h\|_2^2 - \frac{1}{2}\|h(S)\|_2^2 \ge \frac{1}{2}\|h\|_2^2$$

$$2\varepsilon \ge \left[1 - 1/\sqrt{2} - \delta_{3s} - \frac{1}{\sqrt{2}}\delta_{2s}\right]\frac{1}{\sqrt{2}}\|h\|$$