Today: The following are NP-hard:

1. Maximizing a convex function over a convex set.
2. Finding the point of largest norm in a polytope $(Ax \le b)$
3. Computing a matrix p-norm, $\|M\|_p$, $2 < p < \infty$

All will follow from reductions from Max-Cut.

This is a graph problem.

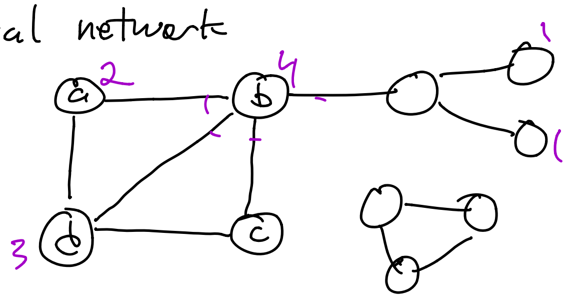
Recall that a graph G, usually written $G = (V, E)$, consists of a set of vertices (aka "nodes"), V, and edges E connecting pairs of nodes.

~~$\{a, b\}$~~

Each edge is a subset of V of size 2, but we write the pair in parentheses, like (a, b)

Ex 1 Social network

Ex 2

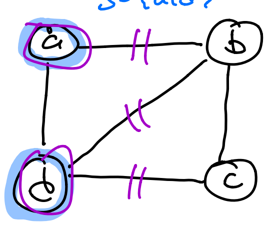


A "cut" is a subset of the vertices, $S \subset V$.

We count the size of the boundary of S, $\partial(S)$

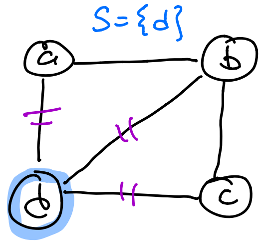
= edges from inside to outside S

$S = \{a, d\}$



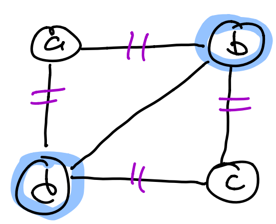
$|\partial(S)| = 3$

$S = \{d\}$



$|\partial(S)| = 3$

$S = \{b, d\}$



$|\partial(S)| = 4$

The maximum cut is $\text{maxcut}(G) = \max_{S \subset V} |\partial(S)|$

To make into a decision problem, we give G and k as input. yes if $\text{maxcut}(G) \geq k$

$(G, k) \in \text{MaxCut}$ if $\exists\, S \subset V$ s.t. $|\partial(S)| \geq k$

answer is "yes" if G has a cut of size $\geq k$

MaxCut is in NP: S is the witness

And, it is NP-complete.

One can not even approximate its value.

→ Håstad '01 proved is NP-hard to even approximate within a factor $18/17$.

That is, for every problem $\Upsilon \in NP$,

$\exists$ a polynomial time algorithm A that outputs G, k

i.e. $A(q) = (G, k)$

s.t. • $q \in \Upsilon \Rightarrow G$ has a cut of size $\geq k$

• $q \notin \Upsilon \Rightarrow$ all cuts in G are smaller than $\frac{17}{18} k$

$\text{maxcut}(G) \leq \frac{17}{18} k$

The result is a little tighter than this: $\frac{17}{16} + \varepsilon$, $\varepsilon > 0$

Is a very deep and complicated result building on a decade of research by many luminaries.

Degree of a vertex is # of attached edges:

$$d_a = |\{b : (a,b) \in E\}|$$

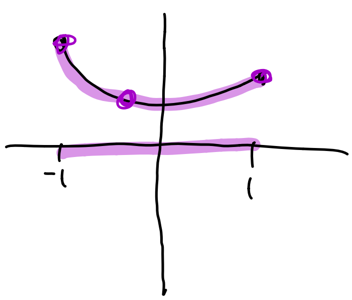
Degree 3-Graphs: Berman & Karpinski '02

reduce to graphs in which every vertex has degree 3 and

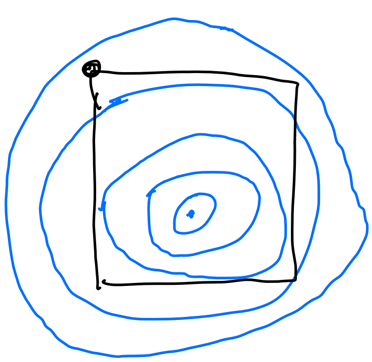
- $q \in Y \Rightarrow \text{maxcut}(G) \geq K$
- $q \notin Y \Rightarrow \text{maxcut}(G) \leq \frac{332}{333} K = \left(1 - \frac{1}{333}\right) K$

Will use this for p-norms.

Maximizing convex functions over convex sets



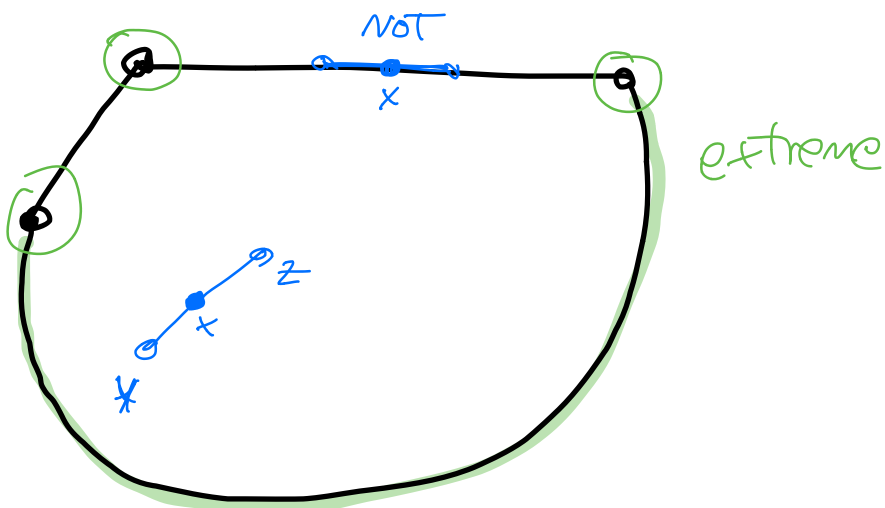
the maximum will be on the boundary, and values inside won't be very helpful.



Claim: the maximum will be at a "corner".

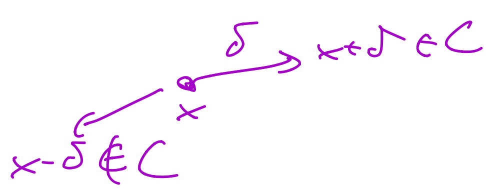
<u>Def</u> x is an <u>extreme point</u> of a convex set C if there do <u>NOT</u> exist $y, z \in C$ and $\lambda \in (0,1)$ s.t.

$$x = \lambda y + (1-\lambda) z.$$



Equivalently, for $\delta \neq 0$, $x+\delta \in C \Rightarrow x-\delta \notin C$.

When C is a polytope like $Ax \le b$, the extreme points are the corners.



Idea of proof for $\{x : \|x\|_\infty \le 1\}$ $-1 \le x(i) \le 1$

let x_* be opt. $y = (1, x_*(2), \ldots, x_*(n))$

$z = (-1, x_*(2), \ldots, x_*(n))$

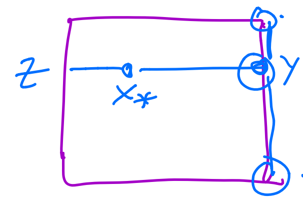
$x_* \in yz$

Convexity $\Rightarrow$ $f(x_*) \le \max(f(y), f(z))$

so go to one of y or z

repeat for other coordinates

eventually make all coords ± 1.



Proof. (for polytopes) let x_* be the maximum.

Carathedory $\Rightarrow$

$\exists$ corners $v_0, \ldots v_d$ s.t. $x_* = \sum_{i=0}^{d} \lambda_i v_i$

$\lambda_i \ge 0$, $\sum \lambda_i = 1$

Convexity $f(x_*) \le \sum \lambda_i f(v_i)$

$\Rightarrow \exists v_i$ s.t. $f(x_*) \le f(v_i)$

A convex relaxation of maxcut.

First, need to write algebraically.

Let $n = |V|$. For a set S, let $x_S(a) = \begin{cases} 1 & a \in S \\ -1 & a \notin S \end{cases}$

For $(a,b) \in E$

$$|x_S(a) - x_S(b)| = \begin{cases} 2 & \text{if } (a,b) \in \partial(S) \\ 0 & \text{o.w., when } x_S(a) = x_S(b) \end{cases}$$

$$\frac{1}{4} \sum_{(a,b) \in E} (x_S(a) - x_S(b))^2 = |\partial(S)|$$

$$Q(x) = \frac{1}{4} \sum_{(a,b) \in E} (x(a) - x(b))^2$$

Is a sum of squares, so is convex.

<u>Claim</u> $\underline{\max_{\|x\|_\infty \le 1} Q(x)} = \underline{\text{maxcut}(G)}$

So, have a polynomial time reduction from maxcut to problem of maximizing Q over $\|x\|_\infty \le 1$, a convex set.

$\Rightarrow$ maximizing convex quadratics over $\|x\|_\infty \le 1$ is NP hard.

The decision problem is: given Q, t, $\exists x : \|x\|_\infty \le 1$, $Q(x) \ge t$?
Is in NP.

Proof $\max Q(x) \geq \text{maxcut}(G)$. $x = x_{S_*}$
where $|\partial(S_*)| = \text{maxcut}(G)$
$\max Q(x) \leq \text{maxcut}$. $\exists x_* \in \{\pm 1\}^n$ s.t. x_* is max
$\exists S$ s.t. $x_* = x_S$
$Q(x_*) = |\partial(S)| \leq \text{maxcut}(G)$

Note: Turned arbitrary maxcut problem
into special convex quadratic.
If don't expect to solve maxcut, shouldn't expect
to maximize convex quadratics.

It is a very special quadratic: the Laplacian
$Q(x) = \frac{1}{4} x^T L x$ where $L(a,b) = \begin{cases} \text{degree of } a & \text{if } a = b \\ -1 & \text{if } (a,b) \in E \\ 0 & \text{o.w.} \end{cases}$

$$(Lx)(a) = d_a x(a) - \sum_{b:(a,b)\in E} x(b)$$

$$x^T L x = \sum_a d_a x(a)^2 - \sum_a \sum_{b:(a,b)\in E} x(a)x(b)$$

$$= \sum_{(a,b)\in E} (x(a)^2 + x(b)^2) - 2\sum_{(a,b)\in E} x(a)x(b)$$

$$= \sum_{(a,b)\in E} (x(a) - x(b))^2 = 4Q(x)$$

$$Q(x) = \frac{1}{4} \sum (x(a) - x(b))^2$$

(9.10)GE

Thm The problem $\max_x \underline{\|x\|_2^2}$ s.t. $\underline{Ax \le b}$ is NP-hard

proof We will reduce the problem

(1) $\max_x x^T L x$ s.t. $\|x\|_\infty \le 1$ to this one

Consider $x^T(L+I)x = x^T L x + \underbrace{x^T x}_{} = n$

(2) $= \max_x x^T(L+I)x$ s.t. $\|x\|_\infty \le 1 = (1) + n$

$L+I$ is pos def $\Rightarrow \exists$ invertible, symmetric B s.t.

$$B^T B = L + I$$

$x^T(L+I)x = x^T B^T B x$

$y = Bx$ and $x = B^{-1} y$

(2) $\Leftrightarrow \max_y \underline{y^T y}$ s.t. $\underline{\|B^{-1} y\|_\infty \le 1}$

$\|B^{-1} y\|_\infty \le 1$ let $b_1 \ldots b_n$ be rows of B^{-1}

get $b_i^T y \le 1 \quad -b_i^T y \le 1$

In fact, hard to approximate.

Matrix Norms for $p > 2$.

$$\|M\|_p = \max_x \frac{\|Mx\|_p}{\|x\|_p} = \max_{\|x\|_p \le 1} \|Mx\|_p$$

$$\|x\|_p = \left(\sum_i x(i)^p \right)^{1/p}$$

$p = 2$ is max singular value

$p = \infty$ is $\max_i \|M(i,\cdot)\|_1$

$p = 1$ is $\max_i \|M(\cdot,i)\|_1$

For all other p is NP-hard.

Bhaskara & Vijayaraghavan show is NP-hard to approximate within any constant.

after we correct minor mistakes

Write $(M, t) \in$ Matrix p-norm if $\|M\|_p \ge t$

Thm For all $p > 2$, $c > 1$, and $Y \in NP$,

$\exists$ poly time algorithm A that outputs M, t s.t.

for $A(q) = (M, t)$

$q \in Y \Rightarrow \|M\|_p \ge ct$

$q \notin Y \Rightarrow \|M\|_p \le t$

A depends upon c.

How prove this for all $c > 1$?

First show for some $c > 1$, and then amplify.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \qquad x \otimes y = \begin{pmatrix} x_1 \cdot y_1 \\ x_1 \cdot y_2 \\ \vdots \\ x_1 \cdot y_n \\ x_2 \cdot y_1 \\ \vdots \\ x_n x_n \end{pmatrix} \in \mathbb{R}^{n^2}$$

$x, y \in \mathbb{R}^n$

$$\begin{pmatrix} x_1 y \\ x_2 y \\ x_3 y \\ \vdots \\ x_n y \end{pmatrix}$$

Recall the Kronecker product:

$$A \otimes B = \begin{pmatrix} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ \vdots & & & \\ a_{n1} B & \cdots & & a_{nn} B \end{pmatrix}$$

$x_i y_j$

$\mathrm{vec}(x y^T)$

Kron

<u>Thm</u> $\|A \otimes B\|_p = \|A\|_p \cdot \|B\|_p$.

proof sketch:

1. For vectors x, y $\quad \|x \otimes y\|_p = \|x\|_p \cdot \|y\|_p$

$$\boxed{(A \otimes B)(x \otimes y) = (Ax) \otimes (By)}$$

$\Rightarrow \|A \otimes B\|_p \geq \|A\|_p \cdot \|B\|_p$

let $x_* \in \arg\max_x \|Ax\|_p / \|x\|_p$, $\quad y_*$ for B

2. $A \otimes B = (A \otimes I)(I \otimes B)$

$\|A \otimes B\|_p \leq \|A \otimes I\|_p \cdot \|I \otimes B\|_p = \|A\|_p \|B\|_p$

$$I \otimes B \qquad \begin{pmatrix} B & & & 0 \\ & B & & \\ & & B & \\ 0 & & \ddots & \\ & & & B \end{pmatrix} \begin{pmatrix} y_1 \\ \\ \\ y_n \end{pmatrix} = \begin{matrix} B y_1 \\ \vdots \\ B y_n \end{matrix}$$

So, if is hard to distinguish $\|M\|_p \le s$ from $\ge b$
is hard to distinguish $\|M \otimes M\|_p \le s^2$ from $\ge b^2$

This squares the inapproximability factor.
Can do this any constant number of times,
and still be a polynomial time reduction.

To get some constant:

Idea: want $\underline{\sum_{(a,b) \in E} |x(a) - x(b)|^p}$ $\quad \|x\|_p \le 1$

But, can't force $x(a) = \pm 1$.

So, add terms to penalize $|x(a)| \notin \underline{(1-\varepsilon, 1+\varepsilon)}$

$\underline{\text{Claim 1}}$: let $F(x) = \dfrac{|x+1|^p + |x-1|^p}{1 + |x|^p}$ $\quad x = -1 \quad \dfrac{0 + 2^p}{1+1}$

a. $F(\pm 1) = \underline{2^{p-1}}$

b. $\forall x,\ F(x) \le \underline{2^{p-1}}$

c. $\underline{\forall \varepsilon > 0}\ \underline{\exists \delta > 0}$ s.t. $|x| \notin (\underline{1-\varepsilon, 1+\varepsilon})$
$\Rightarrow F(x) \le \underline{2^{p-1} - \delta}$

Claim 2: $\forall \varepsilon > 0 \ \exists C > 0$ s.t.

for $$F(x,y) = \frac{|x-y|^p + C\left[|x+1|^p + |x-1|^p + |y+1|^p + |y-1|^p\right]}{2 + |x|^p + |y|^p}$$

if $xy < 0$ and $|x|, |y| \in (1-\varepsilon, 1+\varepsilon)$

$$F(x,y) \le C\,2^{p-1} + \frac{(1+\varepsilon)^p\, 2^{p-1}}{2 + |x|^p + |y|^p}$$

if $xy \ge 0$, $|x| \notin (1-\varepsilon, 1+\varepsilon)$ or $|y| \notin (1-\varepsilon, 1+\varepsilon)$

$$F(x,y) \le C\,2^{p-1} + \frac{(2\varepsilon)^p\, 2^{p-1}}{2 + |x|^p + |y|^p}$$ smaller

For a 3-regular graph G, set

$$g(x) = \frac{\sum_{(a,b)\in E} |x(a) - x(b)|^p + 3C\left[|1+x(a)|^p + |1-x(a)|^p + |1+y(a)|^p + |1-y(a)|^p\right]}{3n + 3\sum_a |x(a)|^p}$$

Lem Let $\text{maxcut}(G) = |\partial(S)| = \gamma \frac{3}{2} n$.

$g(X_S) = C\,2^{p-1} + \gamma 2^{p-2}$, and

$\forall x \quad g(x) \le C\,2^{p-1} + (2\varepsilon)^p 2^{p-2} + \gamma (1+\varepsilon)^p (1-\varepsilon)^p 2^{p-2}$

For any $0 < \gamma_0 < \gamma_1$, is a small enough ε so that computing $\max_x g(x)$ allows one to distinguish

$\text{maxcut}(G) \le \gamma_0 \frac{3}{2} n$ from $\ge \gamma_1 \frac{3}{2} n$

So, is NP-hard to compute $\max_x g(x)$ to within some constant

To turn into a matrix, introduce new variable $x(0)$, which want to have value $n^{1/p}$ so $x(0)/n^{1/p} = 1$, and consider

$$\frac{\displaystyle\sum_{(a,b)\in E} |x(a) - x(b)|^p + 3C\left[\left|\frac{x(0)}{n^{1/p}} - x(a)\right|^p + \left|\frac{x(0)}{n^{1/p}} + x(a)\right|^p + \left|\frac{x(0)}{n^{1/p}} - x(b)\right|^p + \left|\frac{x(0)}{n^{1/p}} - x(b)\right|^p\right]}{3|x(0)|^p + 3\sum_a |x(a)|^p}$$

Show its max value is same as max of g.

Denominator $= 3\|x\|_p^p$ and

Numerator $= \|Mx\|_p^p$ for some matrix M.

So, is hard to compute $\max_x \dfrac{\|Mx\|_p}{\|x\|_p} = \|M\|_p$,

and to approximate it up to some constant.